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ABSTRACT

The purpose of this text is to teach learning and understanding of mathematics at the ninth grade level through the use of science experiments. This text contains significant amounts of material normally found in a beginning algebra class. The material should be found useful for classes in general mathematics as a preparation for enrollment in algebra the following term. Chapters in the text include: (1) An Experimental Approach to the Real Numbers; (2) An Experimental Approach to Linear Functions; (3) The Falling Sphere; (4) An Experimental Approach to Nonlinear Functions; and (5) Analysis of Nonlinear Functions. (RH)

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# MATHEMATICS THROUGH SCIENCE

## PART III: AN EXPERIMENTAL APPROACH TO FUNCTIONS

STUDENT TEXT

(revised edition)

SE 023 019

# MATHEMATICS
# THROUGH SCIENCE
## *Part III: An Experimental Approach to Functions*
## Student Text

(revised edition)

The following is a list of all those who participated in the preparation of this volume:

Elroy J. Bolduc, Jr., The Norwich Free Academy, Norwich, Connecticut
Harry S. Couzins, Alta Loma High School, Alta Loma, California
Bernard Friedman, University of California, Berkeley, California
T. H. Geballe, Bell Telephone Laboratories
A. Lloyd Gelmers, Ponca City Senior High School, Ponca City, Oklahoma
Melvin Greenstadt, Fairfax High School, Los Angeles, California
E. E. Hammond, Jr., Phillips Academy, Andover, Massachusetts
Neal Haygood, Hoffman-Boston High School, Arlington, Virginia
John M. Huffman, Department of Education, San Diego County, San Diego, California
James A. Hummel, University of Maryland, College Park, Maryland
R. R. Kadesch, University of Utah
Frank B. Lindsay, California State Department of Education
Eugene Meyer, S.M., St. Mary's University, San Antonio, Texas
Norman G. Mouck, Jr., Santa Barbara City College, California
William H. Sandman, Harvey Mudd College, Claremont, California
Robert L. Starkey, Cubberley High School, Palo Alto, California
Bryan Thwaites, The University, Southampton, England
Anna Maria Tuske, Jordan Junior High School, Palo Alto, California
James H. Zant, Oklahoma State University

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# PREFACE

Most of the mathematical techniques that are in use today were developed to meet practical needs. The elementary arithmetic operations have obvious uses in everyday life, but the mathematical concepts which are introduced at the junior high school level and above are not as obviously useful.

The School Mathematics Study Group has been exploring the possibility of introducing some of the basic concepts of mathematics through the use of some simple science experiments. Several units were prepared during the summer of 1963 and were used on an experimental basis in a number of classrooms during the following year. On the basis of the results of these trials, these units were revised during the summer of 1964.

This text is designed to be usable with any mathematics textbook in common use. It is not meant to replace the textbook for the course, but to supplement it. Previous acquaintance with science on the part of the student is unnecessary. The scientific principles involved are fairly simple and are explained as much as is necessary in the text. Each experiment opens a door into a new domain in mathematics: measurement, inequalities, the number line, relations and graphs. We hope that student learning and understanding will be improved through the use of this material.

The experiments have all been done in actual classroom situations. Every effort has been made to make the directions for the experiments as clear and simple as possible. The apparatus has been kept to a minimum.

The writers sincerely hope that this approach to mathematics will prove both useful and interesting to the student.

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# Chapter 1

# AN EXPERIMENTAL APPROACH TO THE REAL NUMBERS

## 1.1 Introduction

Much of the development of mathematics has been motivated by the sciences; there was a need to explain and interpret the observations of scientific phenomena. This need continues even today. The mathematics that has been developed to satisfy the needs of the scientist is generally carried far beyond the immediate situation that prompted it. This is the work of the mathematician. These further extensions of mathematics often suggest new theories and experimental possibilities to the scientist.

In short, the connection between mathematics and science is both intimate and far-reaching. Without mathematics the scientist would be unable to systematize and interpret his experiments. He could not generalize his results and make predictions for the outcome of future experiments. Without stimulation from the scientists, mathematicians would work in an unreal world of their own design.

In the work that follows, mathematics will be developed to meet the particular needs of a set of experimental situations. In each case this mathematics will arise from an experimental setting. Once the appropriate mathematical descriptions of the scientific experiment are found, a number of logical extensions of the mathematical structure will be made. In this way we will develop our mathematics in much the same way that mathematics has been developed in the past and continues to be developed today. Although much of the spirit of science will become evident as we proceed, no particular scientific background is required.

## 1.2 The Loaded Beam

Once we have decided to center our investigation upon some particular aspect of nature, we have to make a careful analysis of our proposed experimental procedure to determine the factors that might possibly influence our results. Our first experiment in the physical sciences will involve the bending of a "beam", fixed at one end and loaded at the other. Even a system as simple as this one is susceptible to a wide variety of influences.

The amount of bending will obviously depend on the type of "beam" we choose and the way we load it. Different clamping points for the fixed end and different points for loading the free end will also influence the amount of bending. It is important to permit no more than one of these conditions to influence the bending of the beam at any one time. All other conditions must not be allowed to change.

A 15-inch flexible ruler may be clamped to the desk with a "C-clamp" and used as a beam. (See Figure 1.) As the beam is loaded from the free end, it will bend. To measure the bending of the beam, we will simply record the changing position of the free end of the beam as the load is changed. You may find that some form of a pointer arrangement, such as a straight pin fastened to the free end, will be helpful.

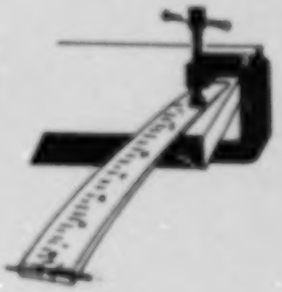


Figure 1

Place a piece of masking tape over the numbers on the meter stick in such a way that the graduations of the scale are not covered (Figure 2). Support the meter stick perpendicular to the floor so that the position of the end of the beam can be read on the scale as the load changes. Adjust the meter stick so that the pointer is in line with one of the centimeter markings on the stick. On the masking tape opposite this mark, write 0. Starting at the 0 mark, draw two arrows along the length of the meter stick, each extending in opposite directions. Label the arrow which points up "up" and label the other arrow "down". Use the counting numbers to number the millimeter graduation on the scale. It is enough to write numbers at 10-mm

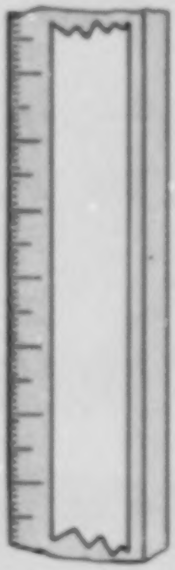


Figure 2

intervals in the upward and downward directions from the 0 as shown in Figure 3.

There should be a small hole in the ruler "beam" about one-half inch from the free end. Thread a piece of nylon string through this hole and around the ruler in such a way that the string is firmly attached to the ruler and about two feet of string hang free on either side of the ruler. (See Figure 4.)

Suspend a single pulley above the beam. A ring-stand or some other similar supporting device may be used for this purpose. Pass one end of the string up over the pulley so that the masses may be hung from the free end of the string. Allow the other end of the string to hang below the end of the beam so that masses may also be hung from this end (Figure 5).

Now hang a 30-gram mass in a downward direction from the load point and take a reading of the position of the end of the beam. Continue by adding 30 grams at a time, until you have at least 10 readings. Be very careful in reading the position of the free end of the beam. Always try to "sight" along the pointer in the same way. Make your position reading to the nearest millimeter.

Remove the load from the beam and repeat the experiment by hanging the masses from the pulley, adding 30 grams each time, until you have at

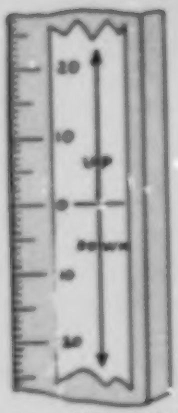


Figure 3

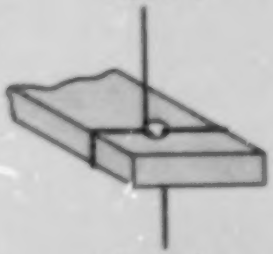


Figure 4

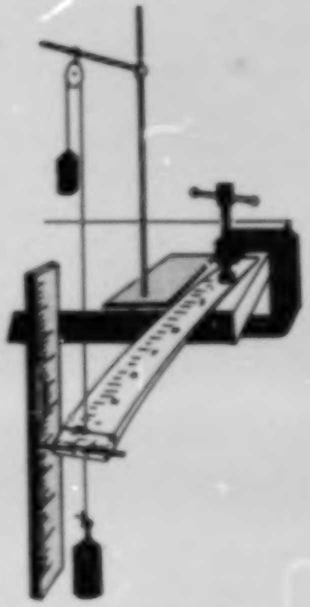


Figure 5

least 10 more readings. Again make your position reading to the nearest millimeter.

You should record your data in an orderly fashion. Along with the load values and the position readings, you should record such things as the type of beam used and its length, i.e., that part which extends outward from the table top to the load point. In recording the position of the end of the beam that is associated with each load, a tabular arrangement will have the most meaning. For example, you could now label your columns for data as shown in Figure 6.

| THE LOADED BEAM EXPERIMENT | | | |
|---|---|---|---|
| Type of Beam____________ | | Length of Beam________ | |
| Load hung down | | Load suspended from pully | |
| Load $\ell$ (grams) | Position $p$ (millimeters) | Load $\ell$ (grams) | Position $p$ (millimeters) |
| | | | |

Figure 6

In this experiment, the beam was deflected both upward and downward. Our scale had been graduated to tell us how much the beam bent, but there was no easy notation to tell the direction of the bending. We could just be careful and always record our reading as "2 mm up" or "6 mm down", but over a large number of readings this notation becomes quite clumsy. But more important than just ease of notation, this idea of direction opens up a new system of numbers which is most useful to the scientist.

If you look at the scale you have made so far, you should note that it is nothing more than a number line on which the numbering extends in either direction from the 0. In the past, when we have made number lines, we indicated that some point on the line was to have a coordinate 0 and some other point was to have the coordinate 1. From this we were able to find points on the line whose coordinates were the counting numbers. We were also able to talk about the numbers assigned as the coordinates of points between any two successive points already located. For example, the point midway between 0 and 1 has the coordinate $\frac{1}{2}$. In our experiment, however, we also proceeded in the opposite direction along this line. This is a portion of the line to which no numbers have as yet been assigned as coordinates. Let us next consider an extension of our set of numbers which will assign numbers as the coordinates of these points.

## 1.3 The Real Number Line

The number line used in the experiment should look similar to the number line in Figure 7. However, the number line does not have to be drawn in a vertical position. In fact, in textbooks, it is most often shown in a horizontal direction.

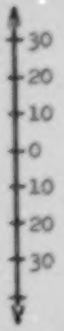


Figure 7

The idea of distance has been a strong underlying theme in the construction of a number line. Let us keep this idea and find a point on this line such that the zero point is just half-way between some new point and the unit point, as in Figure 8.

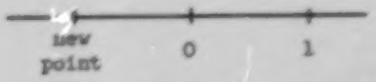


Figure 8

Now the distance from the new point to the zero point is the same as the distance from the zero point to the unit point. In other words, the new point is also one unit of distance from the zero point. However, we have already used the symbol 1 for the coordinate of the unit point and to use the same symbol for the coordinate of a second point would be very confusing. Let us then agree upon a new symbol for the coordinate of this point. The new symbol should tell us that the distance of the point from the zero point is one unit but it is in a direction opposite to that of the unit point. The symbol that has been agreed upon for this coordinate is "-1" (read "negative one").

We might ask in which direction to proceed with the positive numbers and which the negative. This is an arbitrary choice. That is, if we liked, we could start at the zero point and place the unit point to the left. The negative numbers would then be placed to the right of the zero point. However, if we had placed the unit point to the right of the zero point, the whole numbering procedure on the line would have been reversed. Either method is acceptable,but as soon as one method is adopted you should stay with it and be consistent. The method that is used most often is to place the unit point to the right of the zero point.

The set of all numbers used as the coordinates of points on the number line is called the set of real numbers. If the unit point is to the right of zero, then the numbers to the right are called the positive real numbers and the numbers to the left are called the negative real numbers. In this language, the numbers of arithmetic are called the non-negative real numbers.

Each counting number has a negative. There are two points on the number line for each distance. If the distance from zero is to be three units, there is one point 3 units to the right of the zero point, and a second point 3 units to the left. The coordinate of the first point is 3 and the coordinate of the second point is -3. For every number of arithmetic, except zero, the set of real numbers contains a negative number. For example, for every counting number in the set of real numbers, there is also another number in this set which is the negative of the counting number.

The number zero, the counting numbers, and the negatives of the counting numbers all together make up a set of numbers called the integers, {... -3, -2, -1, 0, 1, 2, 3, ...}. Another name for the counting numbers is the positive integers. The set of all negatives of the counting numbers is called the negative integers.

You may recall that a rational number of arithmetic was defined as any number which could be expressed as a fraction having a whole number as a numerator and a counting number as a denominator. Hence, 0, 3, $\frac{1}{4}$, .5 and .33... are all examples of such rational numbers. It is possible to express 0 as $\frac{0}{1}$ and 3 as $\frac{3}{1}$. The set of rational numbers does include the set of all integers. The number $\frac{1}{4}$ is already expressed in the form stated in the definition. It is possible to express .5 as $\frac{1}{2}$, which also satisfies the definition of a rational number. The decimal fraction .33 ... , which is a nonterminating, repeating decimal fraction, can be expressed as $\frac{1}{3}$, and again we see that the definition of a rational number applies to this number.

For all rational numbers, with the exception of zero, there is a negative of that rational number in the set of real numbers. The coordinate of the point midway between -1 and -2 is $-\frac{3}{2}$. The distance of the point to the left of zero is the same as the distance from zero to the point whose coordinate is $\frac{3}{2}$ (Figure 9).

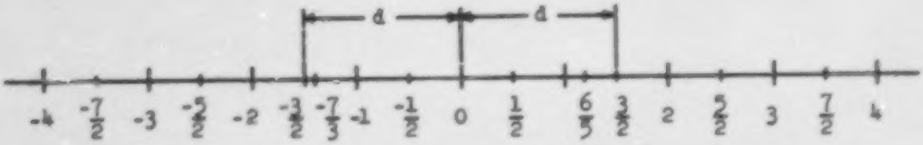


Figure 9

Each rational number is now assigned to a point of the number line, but there remain many points to which rational numbers cannot be assigned. The numbers associated with these points are called the irrational numbers. Irrational numbers are also real numbers. Hence, we can regard the set of real numbers as the combined set of rational and irrational numbers.

Where are some of the points on the number line which do not correspond to rational numbers? There are a great many such numbers, but it is difficult to prove that any particular one is irrational. One example of a real number which can be proved to be irrational is $\sqrt{2}$, that number which when multiplied by itself gives 2. This number is called "the square root of two".

Here is one method of finding a length to which we can associate the square root of two as the measure. This method is based on the understanding of the formulas for the area of a right triangle and the area of a square. You will recall that the area of a right triangle is one-half the product of the lengths of two sides of the triangle which form the right angle. If we have a right triangle such that both of these sides are a unit length (Figure 10) then the area of this triangle is $\frac{1}{2}(1)(1)$ or $\frac{1}{2}$.

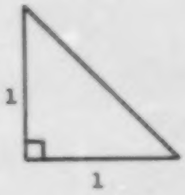


Figure 10

Now if we take four of these triangles and fit them together, like the pieces of a jigsaw puzzle, we notice that they form a square (Figure 11). Since each triangle had an area of $\frac{1}{2}$, and there are four such triangles in the square, the area of the square is $4(\frac{1}{2})$, or 2. The area of a square is the length of a side multiplied by itself.

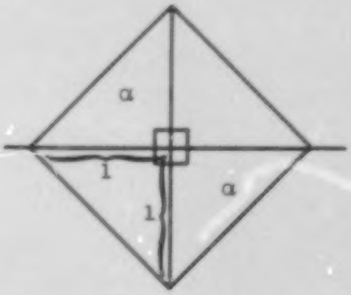


Figure 11

In this case then, the length of the side of the square is the number which, when multiplied by itself, is equal to two. We have already defined this to be $\sqrt{2}$ . In order to locate a point on the number line for $\sqrt{2}$, all we have to do is construct a right triangle with the two sides of the right angle one unit in length and transfer the length of the third side to our number line (Figure 12).

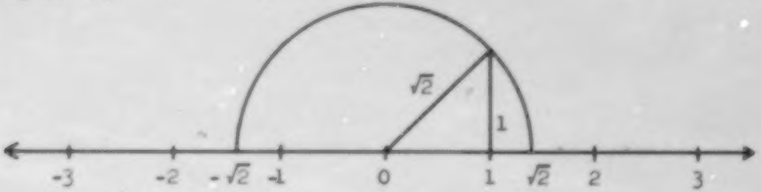


Figure 12

This we can do by drawing a circle whose center is at the point 0 on the number line and whose radius is the same length as the third side of the triangle. This circle cuts the number line in two points, whose coordinates are the real numbers $\sqrt{2}$ and $-\sqrt{2}$ , respectively.

We have avoided trying to prove that the number $\sqrt{2}$ is not a rational number. Such a proof does exist and you will probably study this proof in a later course. At this time, however, you might try to test known rational numbers by squaring them to see if any product is exactly 2. Some numbers you might check are 1.4, 1.41 and 1.4142 .

There are many more points on the real number line which are irrational numbers. Some numbers which serve as coordinates of these points are $\sqrt{3}$, $\pi$ and $-\sqrt{5}$ .

## Exercise 1

1. For each of the following, construct a number line and determine the points whose coordinates are as follows:

   (a) $0, 4, 2, -\frac{1}{2}, -3$

   (b) $\frac{3}{2}, -\frac{3}{2}, 2.5, -2.5, 3$

   (c) $-5, -\frac{5}{2}, \frac{5}{2}, 5, 6$

   (d) $\sqrt{2}, -\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, -2\sqrt{2}$

   (e) $\sqrt{2} + 1, \sqrt{2} - 1, -(\sqrt{2} + 1), -\sqrt{2} + 1$

2. Arrange each set of three numbers given below in the order in which they would appear on the number line, reading from left to right.

   (a) $10, 4, 6$

   (b) $4, 2, -4$

   (c) $-1, -2, -3$

   (d) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

   (e) $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}$

   (f) $\frac{7}{4}, \sqrt{3}, 1.73$

   (g) $-2.24, -\sqrt{5}, -\frac{9}{4}$

   (h) $\sqrt{7}, -2.65, \frac{13}{5}$

3. Which of the following rational numbers is closest to $\sqrt{2}$ ?

   (a) $\frac{3}{2}$

   (b) $\frac{17}{12}$

   (c) $\frac{7}{5}$

   (d) $\frac{99}{70}$

   (e) $\frac{577}{408}$

## 1.4 Ordering the Real Numbers

The number line which we have drawn is a physical model of the set of real numbers. As we continue our discussion of the real numbers, let us recall some of the properties of the positive numbers. These properties have already been well established and we want to make certain that they are not altered in any way.

The first property in which we are interested is the property of order of the positive numbers. We know what is meant by "2 is less then 3". In terms of our physical model, this means that "2 is to the left" of three. In fact, if $a$ and $b$ are two positive numbers on the number line, and $a$ is to the left of $b$, then $a < b$ ($a$ is less than $b$).

Figure 13

As already stated, we are interested in keeping this property over the set of all real numbers. Therefore, our property can now be stated:

> "If $a$ and $b$ are two real numbers on the number line and $a$ is to the left of $b$, then $a < b$."

If we consider the numbers $-1.8$ and $-2.3$ and their relative positions on the number line, we note that the point whose coordinate is $-2.3$ is to the left of the point whose coordinate is $-1.8$. Hence, $-2.3 < -1.8$.

If we are not certain of the position of two numbers on the number line, we can be certain that the following property holds.

> If $a$ and $b$ are real numbers, then exactly one of the following is true:
>
> $a < b$, $a = b$, $b < a$

This property is called the comparison property and holds for all real numbers. For example, if $a = -1.8$ and $b = -2.3$, we have already established that $b < a$.

So far our ordering property has allowed us to compare only two real numbers. To extend our ability to order numbers in sets of three or more, we introduce the transitive property.

> If $a$, $b$ and $c$ are real numbers and if $a < b$ and $b < c$, than $a < c$.

By our definition of the relation $<$, any negative number is less than zero, and zero is less than any positive number. Since we have asserted that the transitive property holds over all real numbers, we now conclude that any negative number is less than any positive number.

The transitive property can also be used in determing the order between two numbers. For example, which is less than the other,

$$\frac{194}{39} \text{ or } \frac{66}{13} ?$$

We notice, after some deliberation, that $\frac{194}{39}$ is approximately 4.97, and $\frac{66}{13}$ is approximately 5.07. Therefore, $\frac{194}{39} < 5$ and $5 < \frac{66}{13}$. From the transitive property it follows that

$$\frac{194}{39} < \frac{66}{13} .$$

We can now use this information to order the numbers $-\frac{194}{39}$ and $-\frac{66}{13}$. Since it has already been established that $\frac{194}{39} < \frac{66}{13}$, we know that $\frac{66}{13}$ is to the right of $\frac{194}{39}$ on the number line. This means that $\frac{66}{13}$ is farther from zero than $\frac{194}{39}$, so $-\frac{66}{13}$ is to the left of $-\frac{194}{39}$ on the number line, and $-\frac{66}{13} < -\frac{194}{39}$.

Our last example shows us that if $a$ and $b$ are both positive numbers and $a < b$, then $-b < -a$ .

## Exercise 2

1. Use appropriate properties to order each of the following pair of numbers.

   (a) $0, 56$

   (b) $0, -7$

   (c) $33.3, 33\frac{1}{3}$

   (d) $-50, -100$

   (e) $\frac{25}{26}, \frac{27}{26}$

   (f) $\frac{2}{3}, \frac{667}{10,000}$

   (g) $-\frac{3}{4}, -.75$

   (h) $-\frac{5}{8}, -\frac{6}{8}$

   (i) $-\pi, -3.14$

   (j) $\sqrt{3}, -1.732$

2. In the blanks below, use one of the symbols =, <, >, to make a true sentence.

   (a) $\frac{3}{5}$ ______ $-\frac{6}{10}$

   (b) $\frac{3}{5}$ ______ $\frac{3}{6}$

   (c) $-\frac{3}{5}$ ______ $-\frac{3}{6}$

   (d) $\sqrt{2}$ ______ $\sqrt{3}$

   (e) $-\frac{1}{3}$ ______ $-.666 \ldots$

   (f) $\frac{9}{16}$ ______ $\frac{1}{2}$

   (g) $\sqrt{2} + 3$ ______ $\sqrt{3} + 2$

   (h) $\frac{1}{4}$ ______ $.125$

   (i) $2\sqrt{5}$ ______ $5\sqrt{2}$

   (j) $-\frac{103}{13}$ ______ $-\frac{205}{26}$

3. Use the transitive property to determine the ordering of the following groups of three real numbers.

   (a) $\frac{1}{5}, \frac{3}{2}, 12$

   (b) $\pi, -\pi, \sqrt{2}$

   (c) $1.7, 0, -1.7$

   (d) $-\frac{27}{15}, -\frac{3}{15}, -\frac{2}{15}$

   (e) $3^2, 4^2, (3 + 4)^2$

   (f) $-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}$

   (g) $1 + \frac{1}{2}, 1 + (\frac{1}{2})^2, 1 + (\frac{1}{2})^3$

4. State a transitive property for "$>$" and illustrate this property with Problems 3(a) and 3(b).

5. Sandy and Bob are seated on opposite ends of a seesaw, and Sandy's end of the seesaw comes slowly to the ground. Harry replaces Sandy at one end of the seesaw, after which Bob's end of the seesaw comes to the ground. Who is heavier, Sandy or Harry?

## 1.5 Opposites

When we loaded the beam from below we noticed that it was deflected downward. The pulley was used when loading the beam from above and this type of loading resulted in a deflection of the beam in the opposite direction. If we refer back to the data we collected in Section 1.2, we can consider the amount of deflection that resulted when equal loads were applied from below and from above. What downward deflection resulted when a mass of 150 grams was hung from the beam? How does this deflection compare with the deflection which resulted when the 150-gram mass pulled up on the beam?

Since we have already developed the concept of negative numbers, we can agree to refer to forces applied in the downward direction (the direction of the earth's gravitational field) as positive and forces applied in an upward direction (against the force of gravity) as negative. The effect of this force was deflection of the beam. In order to be consistent we should use positive numbers to refer to downward bend and negative numbers to refer to upward bend.

The answer to our question might now be: "A load of 150 grams causes the beam to bend 16 mm while a load of -150 grams causes the beam to bend -16 mm." In either case the amount of the deflection is the same. We can think of this as a pairing off of equal distances on the number line from 0 and on opposite sides of 0. Thus, -16 is at the same distance from 0 as is 16. What number is at the same distance from 0 as $\frac{7}{2}$? If you choose any point on the number line, can you find a point at the same distance from 0 and on the opposite side?

Since the two numbers in such pairs are on opposite sides of 0, it is natural to call them opposites. The opposite of a non-zero real number is the other real number, which is at an equal distance from 0 on the real number line (Figure 14). Since there is no other point that is opposite the number 0, we can consider 0 to be its own opposite.

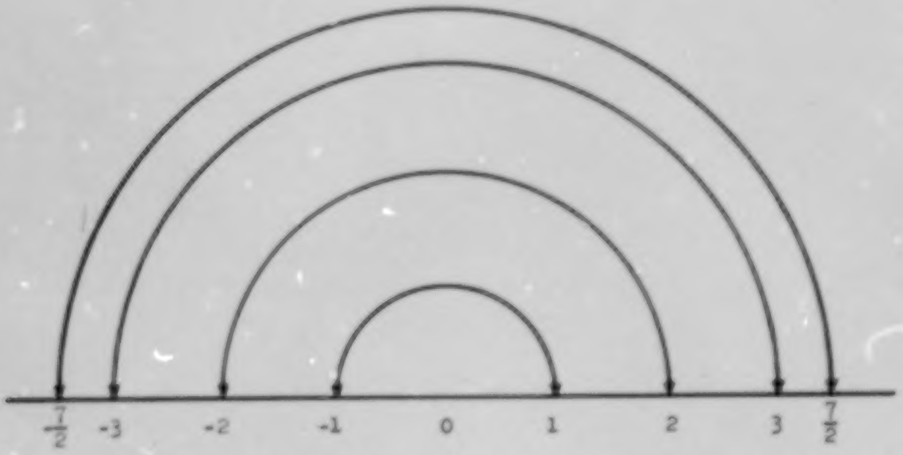


Figure 14

Let us consider some typical real numbers. We have said that -2 is the opposite of 2. What is the opposite of $-\frac{1}{2}$ ? Our discussion leads us to agree that the answer must be $\frac{1}{2}$. However, the notation we have used to indicate that we are referring to the opposite of a number has been to place a negative symbol to the left of the symbol for the number. This means that we might refer to the opposite of $-\frac{1}{2}$ as $-(-\frac{1}{2})$. This last symbol would be read "the negative of a negative one-half". We conclude then that " $\frac{1}{2}$ is the opposite of the opposite of $\frac{1}{2}$ ". This conclusion can be stated in a general way:

> For every real number a, $-(-a) = a$ .

This process of determining the negative of any number can now be used to simplify any notation using the negative of some number. For example, simplify the expression $-[-(-5)]$ . If we consider the number $[-(-5)]$ , our general rule tells us that this number is the same as the number 5. Replacing

$[-(-5)]$ with 5, the problem is reduced to $-[5]$ which, of course, is $-5$ .

In the last section, the statement was made that if a and b are both positive numbers and $a < b$, then $-b < -a$. We can now ask about the ordering of any two real numbers if we know the order of their opposites. A similar argument could be used with two negative numbers. For example, we know that $-10 < -7$ . The opposites of these numbers are 10 and 7. We see that this property holds in this situation since $7 < 10$. There is one other case we should consider. Does this same property apply when one of the numbers is positive and the other negative? Any negative number is less than any positive number, so $-\frac{1}{2} < 2$ . The opposites of these numbers are $\frac{1}{2}$ and $-2$. The order relation between positive and negative numbers still applies; hence, we conclude that $-2 < \frac{1}{2}$ . Again we note that this property for the ordering of opposites holds. The general statement of this property is:

> For real numbers a and b, if $a < b$,
> then $-b < -a$ .

## Exercise 3

1. Simplify each of the following expressions.

   (a) $-(4 + 2)$
   (b) $-(-2.3)$
   (c) $-(42 + 0)$
   (d) $-(3.6) - (2.4)$
   (e) $-(42 \times 0)$
   (f) $-[-(-4)]$
   (g) $-(2 + 5) + 15$
   (h) $-(7 - 10) - 3$
   (i) $-(3 \times 4) + (-3)$
   (j) $-[-(-5)] + 5$
   (k) $-(-7) + [-(-7)]$
   (l) $-(-3) + [-(-3)] - [-(-3)]$

2. What kind of number is $-x$ if $x$ is positive? If $x$ is negative? If $x$ is zero?

3. What kind of number is $x$ if $-x$ is a positive number? If $-x$ is a negative number? If $-x$ is zero?

4. (a) Is every real number the negative of some real number?
   (b) Is the set of all negatives of real numbers the same as the set of all real numbers?
   (c) Is every opposite of a number a negative number?

5. For each of the following pairs, determine which is the greater number.

| | |
|---|---|
| (a) $2.97, -2.97$ | (f) $0.12, 0.24$ |
| (b) $-12, 2$ | (g) $0, -0$ |
| (c) $-358, -762$ | (h) $-0.1, -0.01$ |
| (d) $-1, 1$ | (i) $0.1, 0.01$ |
| (e) $-370, -121$ | |

6. Write true sentences for the following numbers and their opposites, using the relations "$<$" or "$>$".

Example: For the numbers 2 and 7, $2 < 7$, and $-2 > -7$.

(a) $\frac{2}{7}, -\frac{1}{5}$

(b) $\sqrt{2}, -\pi$

(c) $\pi, \frac{22}{7}$

(d) $3(\frac{3}{4} + 2), \frac{5}{4}(20 + 8)$

(e) $-(\frac{8 + 6}{7}), -2$

(f) $-[(3 + 17)0], -[(5 + 0)3]$

7. Let us write "★→" for the phrase "is further from 0 than" on the real number line. Does "★→" have the comparison property enjoyed by "$>$"; that is, if a and b are different real numbers, is it true that a ★→ b or b ★→ a but not both? Does "★→" have a transitive property? For which subset of the set of real numbers do "★→" and "$>$" have the same meaning?

8. Translate the following English sentences into mathematical expressions, describing the variable used:

(a) The load on the beam is greater than 100 grams. What is the load?

(b) The deflection of the beam was no more than 18 mm up. What was the deflection?

(c) Paul hung 30 grams from the beam, but Jim added more than 60 grams to the load. What was the load?

9. Change the numerals "$-\frac{13}{42}$" and "$-\frac{15}{49}$" to forms with the same denominators. (Hint: First do this for $\frac{13}{42}$ and $\frac{15}{49}$.) What is the order of $-\frac{13}{42}$ and $-\frac{15}{49}$?

(Hint: Knowing the order of $\frac{13}{42}$ and $\frac{15}{49}$, what is the order of their opposites?)

Now state a general rule for determining the order of two negative rational numbers.

## 1.6 Absolute Value

Several times our discussion of deflection of the bending beam has referred to the same amount of deflection in either direction. Sometimes it is convenient to consider only the amount with no regard to direction. In order to do this, we want to define a new and very useful operation on a single real number: the operation of taking its absolute value.

> The absolute value of a non-zero real number is the greater of that number and its opposite. The absolute value of 0 is 0.

By this definition we can now state that the absolute value of 4 is 4, because the greater of 4 and -4 is 4. The absolute value of $-\frac{3}{2}$ is $\frac{3}{2}$. (Why?) What is the absolute value of -17? Which is always the greater of a non-zero number and its opposite, the positive or the negative number? We have already established that all positive numbers are greater than any negative number. This then forces the absolute value of any non-zero real number to be a positive number. The symbol we use to indicate the absolute value of a number n is $|n|$. For example,

$$|4| = 4, \quad |-\frac{3}{2}| = \frac{3}{2}, \quad |-\sqrt{2}| = \sqrt{2} .$$

If we look at these numbers on the real number line and consider their absolute values, we can conclude that the distance between a number and zero is the absolute value of the number.

We note that for a non-negative number and zero, the absolute value is the number itself. That is,

For every real number x which is 0 or positive, $(x \geq 0)$,

$$|x| = x .$$

What can be said of a negative number and its absolute value? We have already stated that the opposite of a negative number is greater than the negative number. We also note that this number can be referred to as the negative of the negative number. Our definition of absolute value tells us that $|-5| = -(-5)$ but this number is 5. This leads us to conclude that:

For every negative real number x, $(x < 0)$,

$$|x| = -x .$$

We can now restate the definition of absolute value as follows:

For every positive real number x,

$$|x| = x .$$

For $0 = x$,

$$|x| = x .$$

For every negative real number x,

$$|x| = -x .$$

## Exercise 4

1. Find the absolute values of the following numbers:

(a) $-7$
(b) $-(-3)$
(c) $(6 - 4)$
(d) $-14 + 0$
(e) $-(10 - 8)$
(f) $-(-(-3))$
(g) $14 \times 0$
(h) $(4 + 3) - 7$
(i) $-[-(-5)]$
(j) $-[5(3 - 2)]$

2. For a negative number x, which is greater, x or $|x|$ ?

3. Which of the following statements are true?

(a) $|-7| < 3$
(b) $|-2| \leq |-3|$
(c) $|4| < |1|$
(d) $2 \not< |-3|$
(e) $-3 < 17$
(f) $-2 < -|3|$
(g) $|\sqrt{16}| > |-4|$
(h) $|-2|^2 = 4$

4. Simplify each of the following.

(a) $|2| + |3|$
(b) $|-2| + |3|$
(c) $-(|2| + |3|)$
(d) $-(|-2| + |3|)$
(e) $|-2| - (7 - 5)$
(f) $7 - |-3|$
(g) $|-5| \times 2$
(h) $-(|-5| - 2)$
(i) $|-3| - |2|$
(j) $|-2| + |-3|$
(k) $-(|-3| - 2)$
(l) $-(|-2| + |-3|)$
(m) $3 - |3 - 2|$
(n) $-(|-7| - 6)$
(o) $|-5| \times |-2|$
(p) $-(|-2| \times 5)$
(q) $-(|-5| \times |-2|)$

## 1.7 Addition of Real Numbers

Let us now return to our experiment with the bending beam. We will use the equipment as described in Section 1.2 and illustrated in Figure 5. However, this time we will vary our loading technique and record the results in a slightly different manner.

We will use the same reference which has already been developed in this chapter. All positive loads indicate that the masses are hung from the beam while negative loads indicate that the masses acted on the beam in an upward direction with the aid of the pulley. You should record downward deflection with positive numbers and upward deflection with negative numbers.

We will make five different trial runs under varying load conditions. For Trial I, first load the beam with a 30 gram load and record the deflection reading. Replace the 30 gram load with a 20 gram load, and record this new deflection. Now add the 30 gram load to the 20 gram load and again record the resultant deflection. Now remove these masses from the beam and load it with a 50 gram mass. Figure 15 illustrates a data table similar to the one you should make for recording the results of your experiment.

Trial I

| Load (grams) | Deflection (millimeters) |
|---|---|
| 30 | |
| 20 | |
| 20 + 30 | |
| 50 | |

Figure 15

Now repeat the experiment for Trials II, III, IV and V. Use the loading order indicated in each of the following tables illustrated in Figure 16.

Trial II

| Load (grams) | Deflection (millimeters) |
|---|---|
| -30 | |
| -20 | |
| (-30) + (-20) | |
| 50 | |

Trial III

| Load (grams) | Deflection (millimeters) |
|---|---|
| 200 | |
| -200 | |
| 200 + (-200) | |

Trial IV

| Load (grams) | Deflection (millimeters) |
|---|---|
| 200 | |
| -100 | |
| 200 + (-100) | |

Trial V

| Load (grams) | Deflection (millimeters) |
|---|---|
| -200 | |
| 150 | |
| 150 + (-200) | |
| -50 | |

Figure 16

Let us now consider the results of Trial I. Let us call this table a "physical model" of the "bending beam" since this table helps describe the physical situation which we observed when the beam was loaded and unloaded. From this "physical model" of the experiment we wish to describe a "mathematical model" for the experiment. Our "mathematical model" would give us an accurate description of the experiment if we could be absolutely sure that no errors in measurement were possible and all of the equipment behaved in an "ideal" manner. The "mathematical model" which does satisfy this ideal situation is the operation of addition of the real numbers.

If we take the deflection for 30 grams and add to this the deflection for 20 grams, we get a result which is "ideally" close to the deflection for

50 grams. Suppose a 30-gram load caused a deflection of 3 mm, a 20-gram load a deflection of 2 mm and a 50-gram load a deflection of 5 mm. On the scale we observed that the pointer first went from 0 to 3, and then from 3 it moved two more units in the positive direction. The sum of 3 and 2 is 5. We could also picture this as addition on the number line as illustrated in Figure 17.

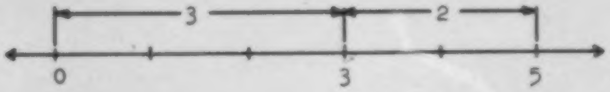


Figure 17

This example reminds us of something we already know: To add a positive number to a positive number, we move to the right on the number line. What happens on the number line when we add a negative number to a negative number?

In Trial II, the loads were all directed up and the deflections were in the same direction. If these deflections are the opposite of those in Trial I, we now observe that (-3) + (-2) = -5 (Figure 18),

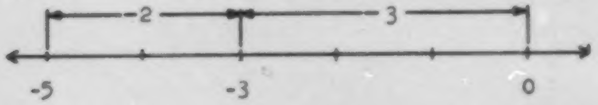


Figure 18

and to add a negative number to a negative number, we move to the left on the number line.

Our next concern is what happens on the number line when we add a positive number and a negative number. Trials III, IV and V give us experimental illustrations of this type of addition.

If our experiment followed the pattern already indicated, Trial III would reveal equal positive and negative deflections. Since positive loads give downward deflection and negative loads upward deflection, we observe that adding a negative load to a positively-loaded beam reverses the direction in which the pointer had moved. In this particular case, the amount of deflection should have been about the same for each individual load. In other words, the two deflections had the same absolute value and the final position on the pointer would be the 0 mark on the scale.

In Trial IV, the positive deflection was greater than the negative deflection. Hence, adding the negative number to the positive number gives a positive result which is equal to the difference between the absolute values of the two deflections. For example, we could consider 20 + (-10). On the number line this could be illustrated by going from 0 to 20 (a distance equal to |20|), then reversing direction and going a distance equal to |10|. This is the same as the operation of subtraction in arithmetic, and what has happened is that we have subtracted 10 from 20 to get a final result of 10.

Trial V is similar to Trial IV, but since the negative deflection was greater than the positive deflection, the final deflection must be negative. Figure 19 illustrates the addition problem -20 + 10 and 10 + (-20). In both cases the final result is -10. Again we notice that the result can be found by taking the difference between |-20| and |10|. However, in this case the negative member has a greater absolute value than the positive number and our "bending beam" model indicates that the end result should be negative.

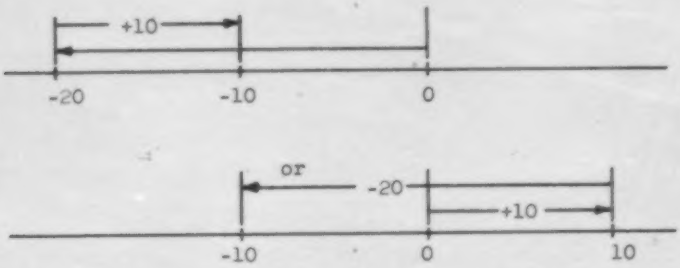


Figure 19

We have now described the motion in all cases. Let us see if we can learn to say how far we move. We want to find the sum of a and b on the number line. First we move |a| units from zero, to the right if a > 0, and to the left if a < 0. From point a we now move |b| units to the right if b > 0 and to the left if b < 0. Check this procedure using the data recorded for Trials I through V (Figures 15 and 16).

When we add two numbers, though, we are not in the habit of using a number line to find the sum. If we add two positive numbers, we merely fall back on our knowledge of the addition facts of arithmetic, i.e., 4 + 6 = 10.

However, what is the sum of two negative numbers? For example, what is

$$(-4) + (-6) \ ?$$

We have found, on the number line, that

$$(-4) + (-6) = (-10) \ .$$

We wish to think a bit more carefully about just how we reached $(-10)$. We begin by moving from 0 to $(-4)$ which is to the left of 0. "Distance between a number and 0" was one of the meanings of the absolute value of a number. Thus, the distance between 0 and $(-4)$ is $|-4|$. Of course we realize that it is easier to write 4 than $|-4|$, but the expression $|-4|$ reminds us that we were thinking of "distance from 0", and this is worth remembering at present. We next add $(-6)$ by moving a distance of $|-6|$ to the left. This results in a new position which is at a distance of 10 units in a negative direction from 0. Hence, $(-4) + (-6) = -(|-4| + |-6|) = -10$ .

You can reasonably ask at this point what we have accomplished by all this. We have taken a simple expression like $(-4) + (-6)$, and made it look more complicated! Yes, but the expression $-(|-4| + |-6|)$, complicated as it looks, has one great advantage. *It contains only operations which we know how to do from previous experience!* Both $|-4|$ and $|-6|$ are positive numbers and we know how to add positive numbers. The sum $-(|-4| + |-6|)$ is the negative of the sum of two positive numbers, and we know how to find that. Thus, we *have* succeeded in expressing the sum of two negative numbers. Prior to this we had just a picture on the number line for this sum.

Think through $(-2) + (-3)$ for yourself, and see that by the same reasoning you arrive at

$$(-2) + (-3) = -(|-2| + |-3|) = -(2 + 3) = -5 \ .$$

From these examples we see that the following defines the sum of two positive numbers in terms of operations which we already know how to do.

> The sum of two negative numbers is negative; the absolute value of this sum is the sum of the absolute values of the numbers.
>
> In general, this statement becomes:
> If a and b are both negative numbers, then
>
> $$a + b = -(|a| + |b|) \ .$$

So far, we have considered the sum of two non-negative numbers, and the sum of two negative numbers. Next we consider the sum of two numbers, one of which is positive, and the other negative.

If we refer back to Trials III, IV and V, we note that the direction of the deflections was reversed. This suggests the operation of finding the difference between two numbers of arithmetic. We can verify that $(-7) + 10 = 10 + (-7) = 3$. We get the same result if we take the difference between the absolute values of these numbers and compare the order of their absolute values. In this case, $|10| > |-7|$ and, therefore, the final result must be a positive number. So,

$$-(7) + 10 = |10| - |-7| = 10 - 7 = 3$$

On the other hand, had we chosen to find the sum, $3 + (-8)$, this is the same as $(-8) + 3$ or $-5$ . Again we have taken the difference between the absolute values of these numbers and compared the order of their absolute values. This final result must be a negative number since $|-8| > |3|$. So,

$$3 + (-8) = -(|-8| - |3|) = -(8 - 3) = -5 .$$

From this it appears that the sum of two numbers, one positive and the other negative is obtained as follows:

The absolute value of the sum is the difference between the absolute values of the numbers.

(a) The sum is 0 if the positive and negative numbers have the same absolute value.

If $a > 0$ and $b < 0$ and $|a| = |b|$, then

$$a + b = (|a| - |b|) = 0 .$$

(b) The sum is positive if the positive number has the greater absolute value. That is, if $a > 0$ and $b < 0$ and $|a| > |b|$, then,

$$a + b = |a| - |b| .$$

(c) The sum is negative if the negative number has the greater absolute value.

If $a > 0$ and $b < 0$ and $|a| < |b|$, then

$$a + b = -(|b| - |a|) .$$

## Exercise 5

1. Perform the indicated additions on real numbers, using the number line to aid you.

   (a) $(-6) + (-7)$
   (b) $(7) + (-6)$
   (c) $(-9) + (5)$
   (d) $6 + (-4)$
   (e) $(-8) + (8)$
   (f) $(25) + (-73)$
   (g) $5\frac{1}{2} + 2\frac{1}{2}$
   (h) $(-2) + (-7)$
   (i) $(-4.6) + (-1.6)$
   (j) $(-3\frac{1}{2}) + (2\frac{2}{3})$

2. Tell in your own words what you do to the two given numbers to find their sum.

   (a) $7 + 10$
   (b) $7 + (-10)$
   (c) $10 + (-7)$
   (d) $(-10) + (-7)$
   (e) $10 + 0$
   (f) $0 + (-7)$
   (g) $7 + |-10|$
   (h) $|7| + (-10)$
   (i) $|7 + (-10)|$
   (j) $(-|-10|) + |7|$

3. In each of the following, find the sum, first according to the definition, and then by any other method you find convenient.

   (a) $(-5) + 3$
   (b) $(-11) + (-5)$
   (c) $(-\frac{8}{3}) + 0$
   (d) $2 + (-2)$
   (e) $18 + (-14)$
   (f) $12 + 7.4$
   (g) $(-\frac{2}{3}) + 5$
   (h) $(-35) + (-65)$

4. In the course of a week the variations in mean temperature from the seasonal normal of 71 were -7, 2, -3, 0, 9, 12, -6. What were the mean temperatures each day? What is the sum of their variations?

### 1.8 The Real Number Plane

We have talked about a coordinate system on a number line, such that every real number is associated with exactly one point of the line. Now let us draw two number lines which are perpendicular to each other. It is not necessary that these two number lines be perpendicular to each other, but this is the type of coordinate system in a plane which we are most likely to see and use. Since the number lines are perpendicular to each other, we will call this a rectangular coordinate system.

We will take the intersection of these two lines as the origin of the coordinate systems of both lines. Each number line is called an axis. We call the axis which extends across the paper the horizontal axis and the other axis the vertical axis. The plane determined by these two axes is called the coordinate plane. Let us agree to place the unit point on the horizontal axis to the right of the origin and the unit point on the vertical axis above the origin. Coordinates may now be assigned to all points on each axis (Figure 20).

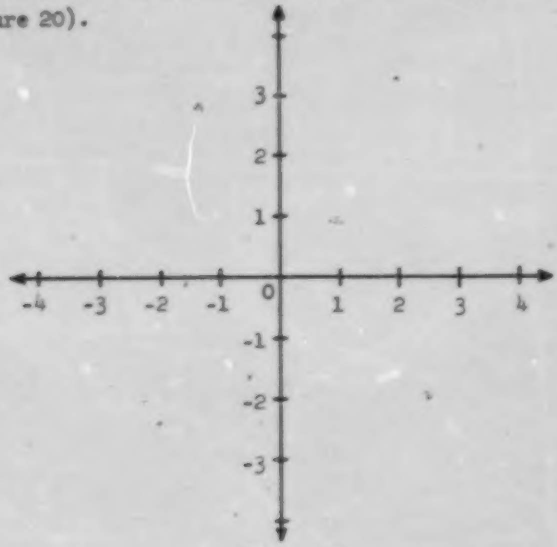
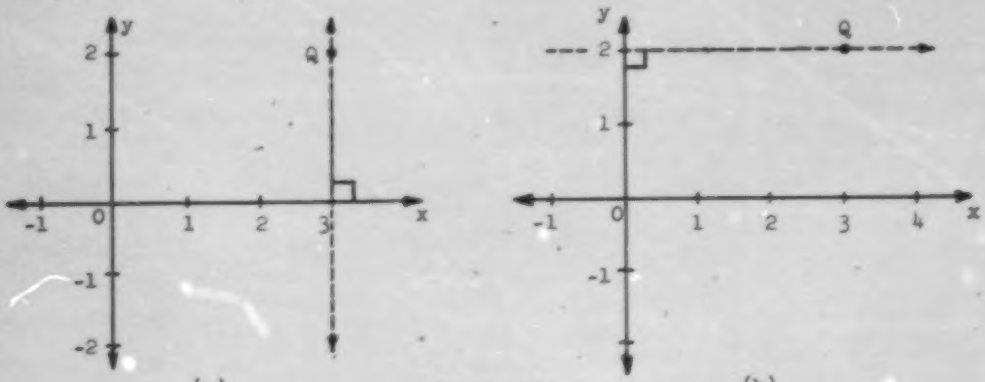


Figure 20

Consider a particular point, such as Q, (Figure 21a) and suppose that a vertical line through Q cuts the horizontal axis in the point whose coordinate is 3.



(a)

Figure 21

(b)

Let us also suppose that a horizontal line through Q cuts the vertical axis in the point whose vertical coordinate is 2 (Figure 21b). We can use this information to define a coordinate system in the coordinate plane. We say that point Q has a horizontal coordinate of 3 and a vertical coordinate of 2. Any point on the coordinate plane can be located if we know its coordinates. Each point has a pair of numbers associated with it. The order in which we write these numbers is important. We write these placing the number found along the horizontal line first, and the one found along the vertical line second and enclosing them in parentheses. We have assigned to Q a first number, 3, and a second number, 2, and we think of these as an ordered pair of numbers, (3,2), belonging to Q and called the coordinates of Q.

In describing the location of a point in the coordinate plane, it is convenient to specify the portion of the plane in which it lies. The horizontal axis and the vertical axis divide the plane into four regions. Each of these regions is called a quadrant. The first quadrant is the set of all points whose horizontal and vertical coordinates are both positive. The second quadrant is the set of all points whose horizontal coordinate is negative and whose vertical coordinate is positive. The third quadrant is the set of all points whose horizontal coordinate and vertical coordinate are both negative. The fourth quadrant is the set of all points whose horizontal coordinate is positive and whose vertical coordinate is negative. We denote these quadrants by I, II, III, IV (Figure 22).

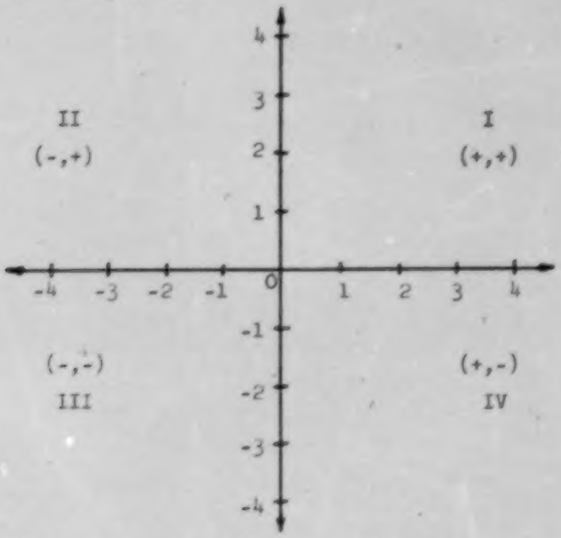


Figure 22

If both of the coordinates are zero, the point is the origin. If the horizontal coordinate is zero and the vertical coordinate is positive, we say that the point is on the positive vertical axis, but if the vertical coordinate is negative, the point is on the negative vertical axis. In a similar manner, if the horizontal coordinate is positive and the vertical coordinate is zero, the point is on the positive horizontal axis; with horizontal coordinate negative, vertical coordinate zero tells us that the point is on the negative horizontal axis.

**Example:** Plot on a coordinate plane the following set of points: {(2,1), (3,-1), (-5,0), (-4,3)}. State the quadrant in which each point falls.

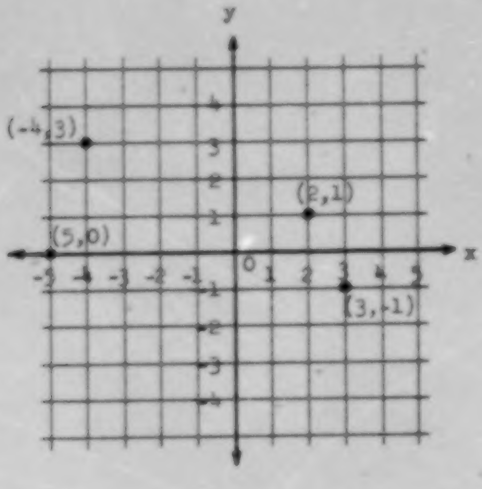


Figure 23

## Exercise 6

1. Plot the following ordered pairs of numbers; write the number of the quadrant or the position on an axis in which you find the point represented by each of these ordered pairs.

(a) (3,5)
(b) (-5,1)
(c) (1,-4)
(d) (-4,4)
(e) (0,0)
(f) (0,5)
(g) (-3,-1)
(h) (7,-1)
(i) (8,6)
(j) (3,-2)
(k) (-3,-5)
(l) (-1,3)
(m) (2,-4)
(n) (5,2)
(o) (-3,0)
(p) (-4,-5)
(q) (-1,2)
(r) (3,-1)

2. (a) Plot on a coordinate plane the following set of points:

$$\{(0,0), (-1,0), (-2,0), (2,0), (-3,0), (3,0)\}.$$

(b) Do all the points in this set seem to lie on the same line?

(c) What do you notice about the vertical coordinate for each of the points?

3. (a) Plot the points in the following set.

$$\{(0,0), (0,-1), (0,1), (0,-2), (0,2), (0,-3), (0,3)\}.$$

(b) Do all the points named in this set seem to be on the same line?

(c) What do you notice about the horizontal coordinate for each of the points?

4. (a) Plot the points in the following set.

$$\{(0,8), (1,6), (2,4), (3,2), (4,0)\}.$$

(b) Do all the points named in this set seem to lie on the same line?

### 1.9 Summary

In this chapter we used a "Loaded Beam" to develop the negative numbers. The experimental results gave us an intuitive understanding of absolute value and the addition of real numbers. The number line was extended to include the negative numbers and used as an aid in addition. The number line was also used to extend the property of ordering for all real numbers.

Finally, we moved from a coordinate system on a number line to the real number plane. A coordinate system for the plane was developed and we learned to associate ordered pairs of numbers with points on the plane.

Chapter 2

# AN EXPERIMENTAL APPROACH TO LINEAR FUNCTIONS

## 2.1 Real Number Generator

The previous experiment with the loaded beam made it possible to "generate" the real numbers. Now we want to construct a "simple" real number generator which we can use to demonstrate more properties of the real number system. This device will make it possible to "turn" a real number.

Take a one-foot piece of $\frac{1}{2}$ - inch threaded rod with a fitting hex nut and washer. Glue the washer to the hex nut and thread the combination on the rod. Support the rod with two transparent tape holders and modeling clay, as shown in Figure 1.

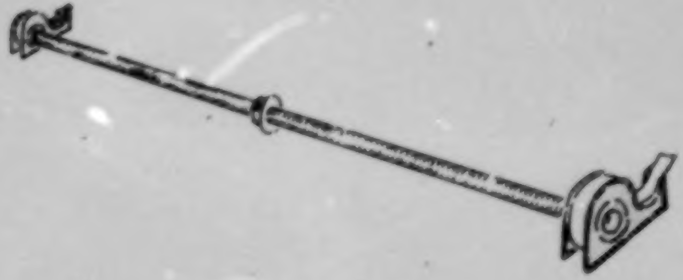
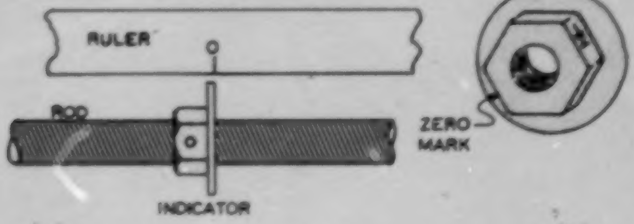


Figure 1

The "indicator" is the washer glued to the hex nut. Place masking tape on the faces of the hex nut. Move the indicator by rotating it until it is in the approximate center of the ruler and one face of the hex nut is in a level position. Mark this face of the nut with the numeral zero. Make a mark on the ruler opposite the edge of the washer and label it with the numeral zero also. The mark on the ruler should be located even with the plain face of the nut-washer combination (Figure 2a). We have arbitrarily chosen both the point and the face for zeros.



(a) (b)

Figure 2

Now rotate the indicator halfway (to the third face following the one marked zero) and mark this face $\frac{1}{2}$ (Figure 2b). Now as you rotate the indicator you can see that each face represents $\frac{1}{6}$ of a turn. Each full rotation of the indicator could be chosen as our arbitrary unit of displacement. However, this unit would be very small so let us choose ten turns to equal one unit. This will give us a decimal system similar to our monetary system and the metric system. Since there are six faces in each turn and ten turns to the unit, there are sixty faces to one unit.

$$\boxed{\begin{array}{l} \left(\frac{6 \text{ faces}}{1 \text{ turn}}\right) \times \left(\frac{10 \text{ turns}}{1 \text{ unit}}\right) = \\ \left(\frac{6 \text{ faces} \times 10 \text{ turns}}{1 \text{ turn} \times 1 \text{ unit}}\right) = \\ \left(\frac{6 \text{ faces}}{1 \text{ unit}}\right) \times \left(\frac{10 \text{ turns}}{1 \text{ turn}}\right) = \\ \left(\frac{6 \text{ faces}}{1 \text{ unit}}\right) \times (10) = \frac{60 \text{ faces}}{1 \text{ unit}} \end{array}}$$

This should remind us of the way in which we count time by sixties (i.e., sixty seconds is one minute or sixty minutes is one hour).

Begin at zero and move the indicator to the right. Each time you complete ten turns and the face of the hex nut marked zero is on top, mark the ruler as before. After you reach the support at the end return to zero and move the indicator in the same manner to the left. After you have marked off units to the left of zero, return the indicator to zero. Now label the marks to the right of zero with positive integers (1, 2, 3, ...) and the marks to the left of zero with the negative integers (-1, -2, -3, ...). We now have the scale marked with the integers as in Figure 3 (... -3, -2, -1, 0, 1, 2, 3, ...) .

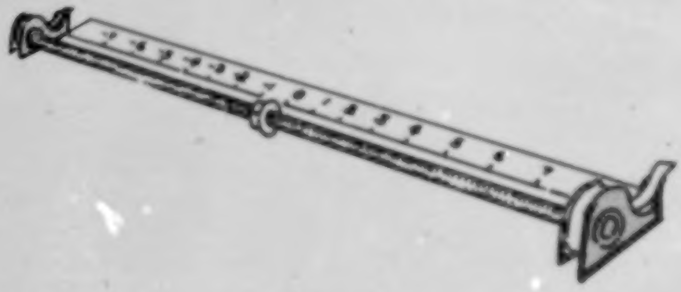


Figure 3

# ERRATA SHEET

Mathematics Through Science Part III. Student Text (revised)

(The following material was omitted from the text. It follows page 30.)

The marked ruler is the physical model of our number line. The indicator on the threaded rod makes it possible to "turn" or generate numbers. Since ten turns of the indicator generates our unit, one unit to the right is 1 and one unit to the left is -1. Then one turn of the indicator to the right generates 0.1 or $\frac{1}{10}$, which is a positive rational number. One-half turn on the indicator to the left from zero generates -0.05 or $-\frac{1}{20}$, which is a negative rational number. A sixth of a turn (one face on the unit) to the right of zero will generate $\frac{1}{60}$ or 0.01666.... .

Turn the indicator $12\frac{2}{3}$ turns to the right and find what rational number is generated. We discover that it is between 1 (ten turns) and 2 (twenty turns). Since two turns is 0.2 and $\frac{2}{3}\ \frac{6 \text{ faces}}{1 \text{ turn}}$ is 4 faces, our rational number could be represented by $1 + 1/5 + 4(1/60)$ or $1 + 0.2 + 4(0.01666...)$. This same number can be represented as $\frac{19}{15}$

(Check the arithmetic yourself.)

## Exercise 1

1. How many turns of the indicator are necessary to generate the following numbers?

a) 3
b) -4
c) 1.4
d) -2.8
e) 5.45
f) $-1\frac{1}{3}$

2. How many face changes of the hex nut from the zero point will generate the following numbers?

a) 2
b) -3
c) $\frac{33}{60}$ or $\frac{11}{20}$
e) $\frac{7}{12}$
f) $-4\frac{11}{15}$
g) $\frac{3}{5}$
h) $\frac{-15}{3}$
i) $\frac{27}{6}$
j) $\frac{5}{4}$
k) $5\frac{1}{3}$
l) $7\frac{3}{10}$
m) $-2\frac{1}{10}$
n) $\frac{-5}{6}$
o) $-\frac{5}{12}$

3. What numbers would be generated by the following number of turns of the indicator?

a) Right 35
b) Left 15
c) Right 95
d) Left 42
e) Right $17\frac{1}{2}$
f) Left $2\frac{1}{3}$

4. What numbers would be generated by the following number of face changes from the zero position?

(a) Right 90
(b) Left 45
(c) Left 256
(d) Right 156
(e) Left 512
(f) Right 316

## 2.2 Functions and Relations

We can continue to perform the experiment. There is a correspondence between the numbers on the scale and either the number of turns of the indicator or the number of face changes. Let us consider a set of ordered pairs such that any first element of an ordered pair is the displacement from the zero point, and any second element in an ordered pair is the number of turns of the indicator which generated the displacement. Let S represent an element of the first set, and T an element of the second set. The ordered pairs (S,T) represent our correspondence. The zero point on the scale corresponds to the zero mark on the indicator and (0,0) represents this correspondence. We already have many other ordered pairs from the previous discussion, such as (1,10), (-1,10), $(\frac{1}{20}, \frac{1}{2})$ and $(\frac{19}{15}, \frac{38}{3})$ . Find six other ordered pairs and plot them all on coordinate graph paper. Label the horizontal axis, S, and the vertical axis, T.

You will notice that the set of points plotted on the graph paper are arranged in such a way that they suggest straight lines. In fact, they appear to be a pair of straight lines meeting at the origin. Draw the lines that best fit your data. Now we have another physical model of our correspondence.

Once we have decided to depart from the experimental "facts" and draw straight lines to represent our data, we have a graph similar to Figure 4.

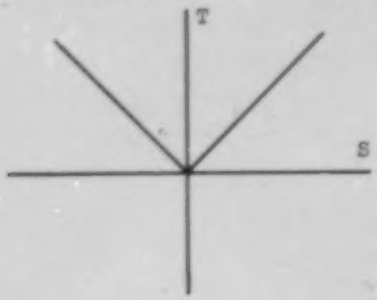


Figure 4

Such a graph exhibits a "relation" between the numbers S and the number of turns T. In all relations there are three essential features: a domain, a range and some "rule" which will tell us when an element of the domain and an element of the range satisfy the relation. This "rule" does not have to be a simple algebraic expression. It sometimes can be only a list of arbitrary pairings with no underlying pattern. In fact, we *define* a *relation* to be a set of ordered pairs. The set of all the *first elements* in the ordered pairs of the relation is called the *domain* of the relation and the set of all the *second elements* is called the *range*.

In an experimental setting, the domain is usually limited by the physical arrangement of the experiment, e.g., in the previous chapter the amount of "load" we could hang on the beam and in this experiment by the length of the threaded rod. Once the domain had been fixed, we then determine the range experimentally. With the aid of a graph we can search for a connecting link between the domain and the range. If we can find such a connection, we may be able to use it to "generate" elements of the range which correspond to given elements in the domain.

In the graph of the relation which we found in the number generator experiment (Figure 4) we can see an important feature. Each element of the domain has associated with it exactly one element of the range. To be specific, for every number on the scale there is only one number representing the number of turns of the indicator. This type of relation is of special importance in mathematics. It is called a *function*. A relation is a *function* if for each first element in the ordered pairs there exists exactly one second element.

Our investigation has shown us that what we really have is a functional relation whose domain is all numbers from the scale used in the experiment, and whose range is the number of turns from zero. When we draw a continuous line on our graph, we are expressing the idea of continuity of the relation.

Now the question is whether or not the graph is actually a representation of the functional relation. We can answer the question in the affirmative if we can satisfy ourselves on just one more point. Does the graph show that each element in the domain has exactly one element in the range related to it?

To answer this question, we must learn how to find the related elements of the range if we are given elements of the domain. We shall illustrate this procedure with an example. Figure 5 shows a straight line graph.

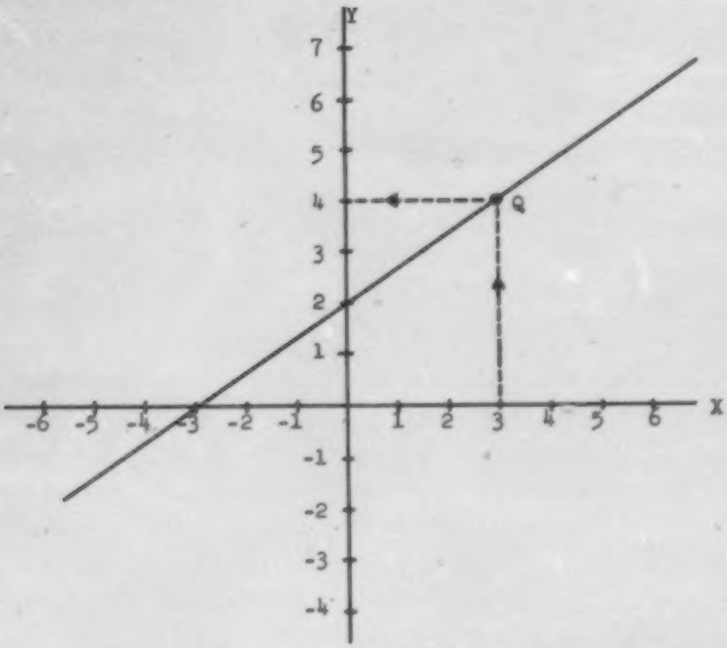


Figure 5

As you know, any point on this line can be represented by an ordered pair (x,y) which expresses the horizontal and vertical coordinates, respectively. of that point. We take some point on the x-axis and from this point we draw (or imagine) a vertical line. We now examine this line to determine how many intersections it has with the graph. At each point of intersection with the graph we have an ordered pair of the relation. If the line intersects the graph in exactly one point, we know that that particular element in the domain has exactly one element in the range related to it. If every possible vertical line drawn from the elements in the domain intersects the graph in exactly one point, then every element in the domain has exactly one element in the range related to it and this relation is a function.

Suppose we are given the point whose x-coordinate is 3 and we wish to find the related y-coordinate. Draw the vertical line whose x-coordinate is 3 and consider the point at which this line intersects the graph. From this point of intersection we draw a horizontal line, extending it, in turn, until it meets the y-axis. The value of y at this last point is the value related to x = 3. The sequence of steps is shown in Figure 5 by dashed lines and arrowheads. We note from Figure 5 that for x = 3 we get y = 4. Thus, the ordered pair (represented by the point Q in the figure) is (3,4) and we have found the unique element of the range which is related to a given element of the domain.

Now let us apply this procedure in getting an answer to the questions we posed. Is the graph of Figure 4 the graph of a function? The elements of the domain of our scale-turns relation are plotted on the S axis (horizontal) and the elements of the range on the T axis (vertical). Start with any point on the horizontal axis and from it draw a vertical line to the graph. From the point of intersection draw a horizontal line to the vertical axis. Clearly, any vertical line you could draw intersects the graph at only one point and, therefore, any value in the domain has related to it a single value in the range. Thus, the graph represents the functional relation. (See Figure 6.)

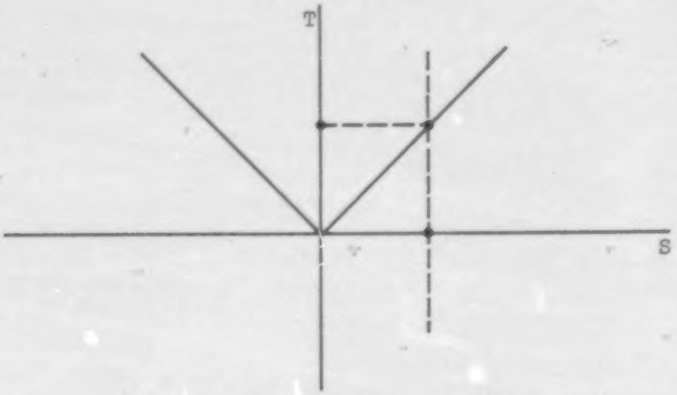


Figure 6

## 2.3 The Face-Scale Relation

Let us collect another set of ordered pairs from the number generator. For this set of ordered pairs, we will let the first element be the number of face changes from the zero position and the second element be the corresponding number on the scale. Let us represent such an ordered pair by (F,S) where F is the number of face changes, and S is the number on the scale. The ordered pair (0,0) and the pair (60,1) are both in this set of ordered pairs. Find eight members of this new set. Graph this set of ordered pairs. Label the horizontal axis F and the vertical axis S. Your graph will look something like Figure 7.

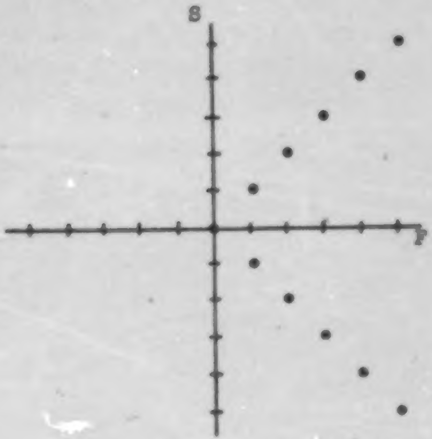


Figure 7

Just as before, we can connect these points by straight lines. Every real number S can be obtained by some (possibly fractional) number of face changes. Thus, the graph of this relation will be similar to the graph shown in Figure 8. Is this the graph of a function? Let us apply the test discussed in the last section and see. Start with any point in the domain and draw a vertical line to the graph.

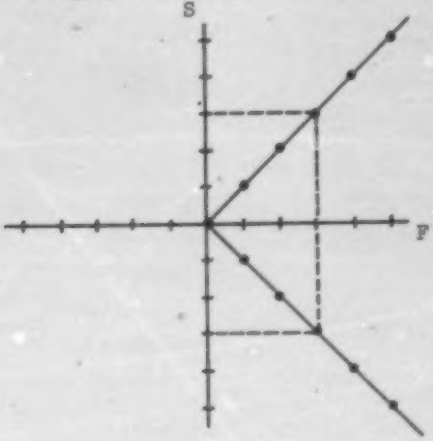


Figure 8

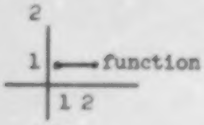
You find two points of intersection and now you must draw two horizontal lines to the vertical axis. This means that this value in the domain has two values in the range related with it. Thus, the graph of this relation does not represent a function.

To summarize, the experiment shows that the numbers on the scale and number of turns of the threaded rod form a functional relation. Moreover, the domain includes all numbers between the largest and smallest marked on the ruler. The range includes zero and all positive numbers. There are no breaks in the graph; hence, we say the function is continuous. The number of face changes from a fixed position and numbers on the scale, form a relation which is not a function.

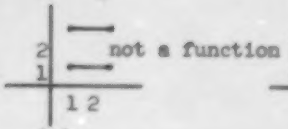
## Exercise 2

1. Which of the graphs of the relations shown below is the graph of a function?

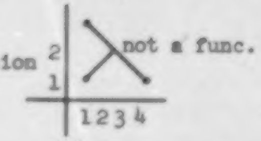
Examples:



(1)

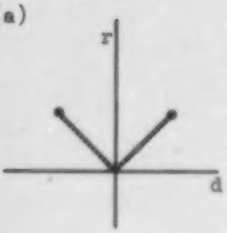


(2)

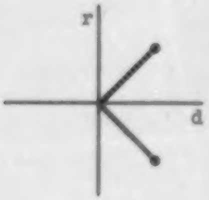


(3)

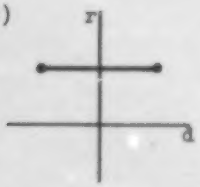
(a)



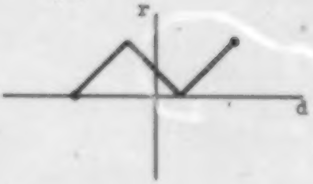
(b)



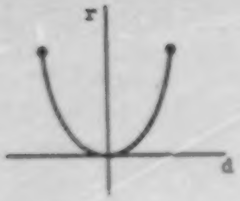
(c)



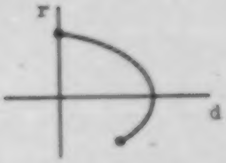
(d)



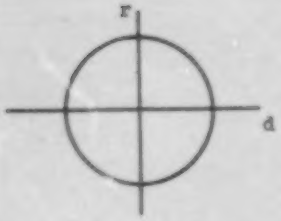
(e)



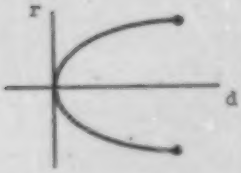
(f)



(g)

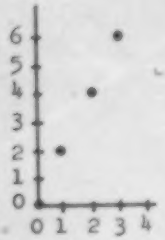


(h)



2. Graph the ordered pairs given below, state the domain and range, and tell if the relation is a function.

Example: {(0,0), (1,2), (2,4), (3,6)}



Domain {0, 1, 2, 3}
Range {0, 2, 4, 6}
Relation is a function
(discrete)

(a) {(1,2), (-1,2), (-2,4), (2,4)}
(b) {(1,3), (1,-3), (3,9), (3,-9)}
(c) {(-1,-2), (-1,2), (-4,-6), (-4,6)}
(d) $\{(\frac{1}{2}, \frac{3}{4}), (\frac{1}{2},-\frac{3}{4}), (\frac{3}{2}, \frac{7}{4}), (\frac{3}{2},-\frac{7}{4})\}$
(e) $\{(\frac{1}{4}, 5), (-\frac{1}{4}, 5), (\frac{1}{2},10), (-\frac{1}{2}, 10)\}$

## 2.4 Seesaw Experiment and Multiplication of Numbers

In Section 1.7 you studied the addition of real numbers and learned the meaning of such phrases as $(-3) + (-2)$ and $(2) + (-4)$. In the next few sections we will be faced with the problem of multiplication. All we can say at present is that we know how to multiply two non-negative numbers.

We will use a simple "seesaw" to illustrate multiplication and to provide us with an intuitive feeling for such products as

$$(-2)(3); \quad (2)(-3); \quad (-2)(-3) .$$

Our experimental setup is shown in Figure 9.

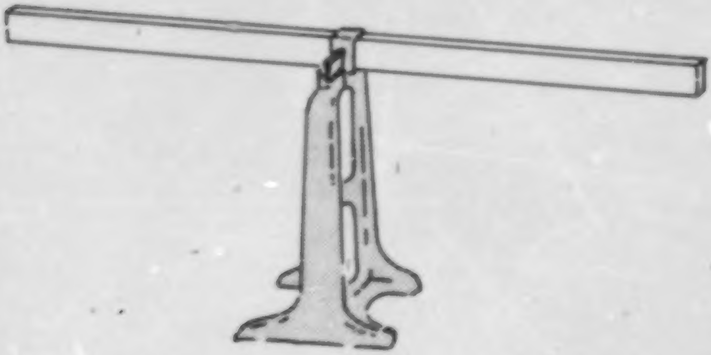


Figure 9

The apparatus consists of a balance support, a knife edge clamp, and a meter stick. You may be familiar with this type of equipment as it is used in a science classroom in studying equilibrium and lever action. The knife edge clamp acts as a point of rotation (fulcrum). The knife edge should be adjusted on the meter stick so that the stick balances in a horizontal position. The same arrangement can be constructed by using a triangular block of wood as a support for the meter stick.

If a force is applied to either arm, the meter stick will begin to tip. Force can be applied by hanging a weight from the meter stick or using a pulley arrangement to change the direction of the force. The sketch below shows a set of forces acting on a balanced meter stick (Figure 10).

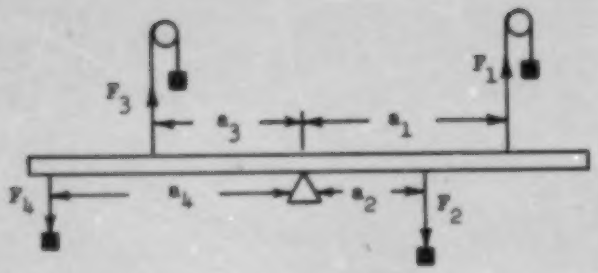


Figure 10

A force like $F_2$ or $F_3$ tends to rotate the lever about the fulcrum in a clockwise direction. $F_1$ and $F_4$, however, tend to rotate it in the counter-clockwise direction.

The sense of rotation connected with a force and lever can be used to develop the multiplication of real numbers. Together, the lever and force form a "force multiplier". A given force can accomplish a great deal if it is applied through a long arm. A pipe wrench is a familiar example. One does not expect to be able to tighten a nut with his fingers. The wrench provides a lever arm through which the force can act; it multiplies the effect of the force. Following this reasoning we define the moment of a force as a product:

$$a\ (\text{arm}) \times F\ (\text{force}) = L\ (\text{moment of force})$$

where a is the distance between the point of application of the force and the fulcrum.

If we suspend a one-pound weight at a distance of two feet from the fulcrum, the turning moment is 2 ft-lb. The units connected with moment of force are formed by taking the product of the force units and length units. This procedure is not new. We are familiar with the process of "multiplying" two lengths to form length squared which is the unit of area. In our present system the unit of moment of force is the foot-pound (ft-lb).

Clearly, the sense of the moment does not depend on the weight used. Also, changing the length of the arm will not change the sense of the moment. To alter the sense of the moment, we must reverse the direction of the force (by pulling up on the arm, or by introducing a pulley) or apply the force from the opposite side of the fulcrum. To take account of these considerations, we can make an analogy between forces and arms and the vertical and horizontal axes of the coordinate plane (Figure 12).

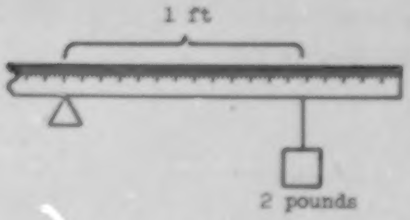


Figure 11

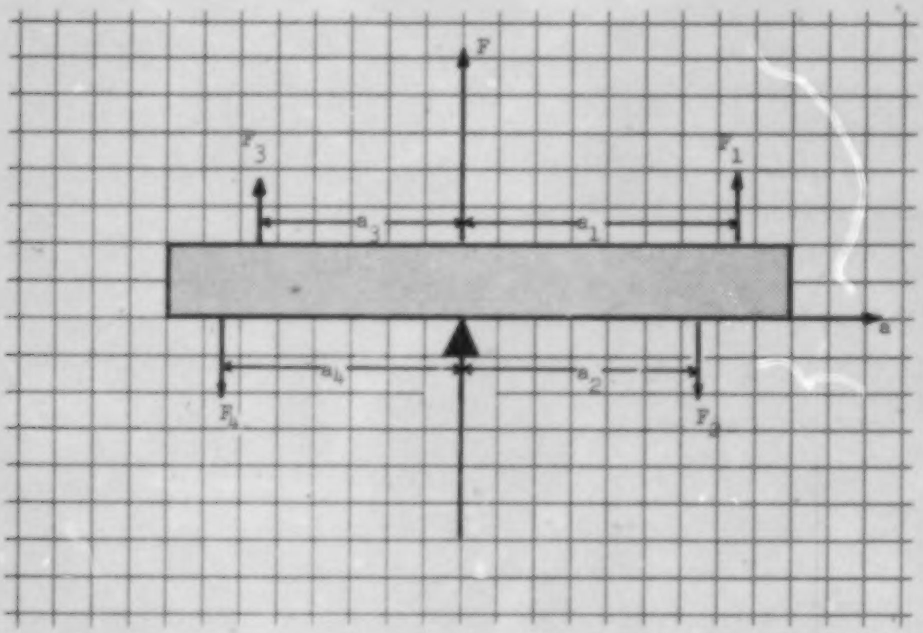


Figure 12

If we use the same sign convention used in plotting points, upward forces like $F_1$ and $F_3$ will be positive, and those acting in a downward direction like $F_2$ and $F_4$ will be negative. Again, as in drawing graphs, we consider distances to the right of the origin (fulcrum) to be positive, and distances to the left of the origin to be negative. Thus, the arms $a_1$ and $a_2$ are positive, while $a_3$ and $a_4$ are negative.

In Figure 12 a force, $F_1$, acts upwards (positive) at a distance $a_1$ to the right of the fulcrum (positive). This has a tendency to give the lever a counterclockwise rotation. The product of a *positive* force and a *positive* arm should give a *positive* number which we call a positive moment.

$$(+)\cdot(+)=(+)\quad .$$

If we now consider $F_4$ we see that it is a *negative* force, and its arm $a_4$ is also *negative*. The effect produced by this combination is, however, a counterclockwise rotation which we have just called a positive moment. Thus,

$$(-)\cdot(-)=(+)\quad .$$

This last statement, if it can be applied to numbers, indicates that the product of two negative numbers is a positive number. You must remember, however, that this is not a proof for a statement about numbers. We have only shown that we can find an intuitive interpretation for the product of two negatives.

It is left to the student to satisfy himself that consideration of $F_2$ and $a_2$ gives

$$(-)\cdot(+)=(-)$$

and of $F_3$ and $a_3$ gives

$$(+)\cdot(-)=(-)\quad .$$

Our sign convention for the seesaw can be summarized by the following figure.

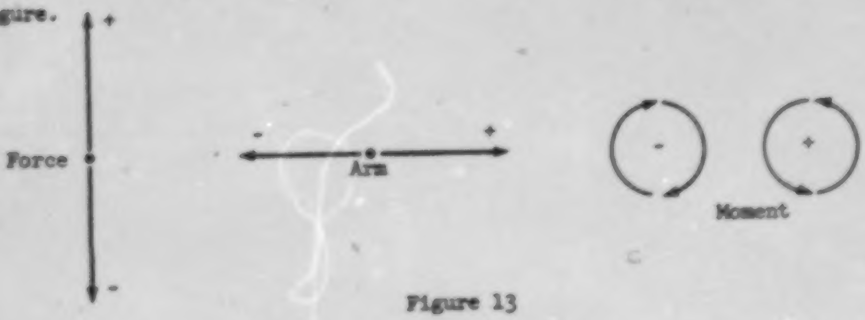


Figure 13

Now let us try to see if these same results can be obtained by using some of the facts we know about numbers. If a, b and c are any numbers of arithmetic, then

$$(a)(b) = (b)(a)$$
$$(a)(1) = (a)$$
$$(a)(0) = (0)$$
$$(a)(b + c) = ab + ac$$

Whatever meaning we give to the product of two real numbers must agree with the products which we already have for non-negative real numbers. The above properties of multiplication which held for the numbers of arithmetic must still hold for all real numbers. We can test the product of a positive number and a negative number with the following example:

| | |
|---|---|
| $0 = (3)(0)$ | |
| $0 = (3)(2 + (-2))$ | by writing $0 = 2 + (-2)$. (Notice how this number introduces a negative number into the discussion.) |
| $0 = (3)(2) + (3)(-2)$ | if the distributive property is to hold for real numbers. |
| $0 = 6 + (3)(-2)$ | since $(3)(2) = 6$ . |

We know from our study of addition that the number which yields 0 when added to 6 is the number -6. Therefore, if the properties of numbers are expected to hold, (3)(-2) must be equal to -6.

Next, we take a similar course to investigate the product of two negative numbers.

| | |
|---|---|
| $0 = (-2)(0)$ | if the multiplication property of 0 is to hold for real numbers. |
| $0 = (-2)(3 + (-3))$ | by writing $0 = 3 + (-3)$. |
| $0 = (-2)(3) + (-2)(-3)$ | if the distributive property is to hold for real numbers. |
| $0 = (-6) + (-2)(-3)$ | if the commutative property is to hold for real numbers, then $(-2)(3) = (3)(-2)$. But the result of the previous problem was $(3)(-2) = -6$ . |

Now we have come to a point where (-2)(-3) must be the opposite of -6. The number which must be added to -6 to yield zero is the number 6. Hence, if we want the properties of multiplication to hold for real numbers, then (-2)(-3) must be 6. Could the same argument be used with any pair of negative numbers?

## Exercise 3

1. Fill in the blanks:

(a) The product of two positive numbers is a _______ number.

(b) The product of two negative numbers is a _______ number.

(c) The product of a negative and a positive number is a _______ number.

(d) The product of a real number and 0 is _______ .

2. Calculate the following:

(a) $(-\frac{1}{2})(-4)$

(b) $((-\frac{1}{2})(2))\ (-5)$

(c) $(-\frac{1}{2})((2)(-5))$

(d) $(-3)(-4) + (-3)(7)$

(e) $(-3)\ ((-4) + 7)$

(f) $(-3)(-4) + 7$

(g) $|-3|(-4) + 7$

(h) $|3||-2| + (-6)$

(i) $(-3)(|-2| + (-6))$

(j) $(-3)(|-2| + (-6))$

(k) $(-0.5)\ (|-1.5| + (-4.2))$

3. Find the values of the following for $x = -2$, $y = 3$, $a = -4$:

(a) $2x + 7y$

(b) $3(-x) + ((-4)y + 7(-a))$

(c) $x^2 + 2(xa) + a^2$

(d) $(x + a)^2$

(e) $x^2 + (3|a| + (-4)|y|)$

(f) $|x + 2| + (-5)\ |(-3) + 2|$

### 2.5 Slope

You may recall from your study of the number line that the distance from one point to another is the coordinate of the one point minus the coordinate of the other.

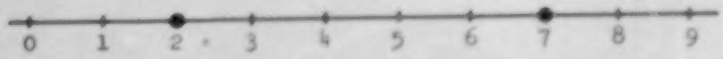


Figure 14

For example, the distance between the points whose coordinates are 2 and 7 is 7 - 2 or 5 (Figure 14). If the points are not on the number line, but are points on the coordinate plane, the question of finding the distance between these points becomes much more complicated. There are some cases, however, which are not too difficult to determine.

A horizontal line in the coordinate plane is defined as a line whose points are ordered pairs with the same second element. The line illustrated in Figure 15 is an example of a horizontal line. What is the distance between the two points whose coordinates are (3,2) and (7,2)? Let us define the distance on a horizontal line as the first element of one ordered pair subtracted from the first element of the other ordered pair; that is, 7 - 3. Therefore, in this example, the distance between the two points is 4.

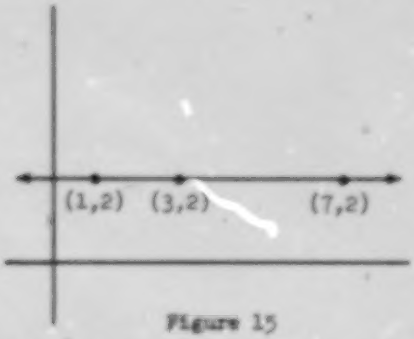


Figure 15

A vertical line in the coordinate plane is defined as a line whose points are ordered pairs with the same first element. The distance between any two points on a vertical line is the second element of one ordered pair subtracted from the second element of the other ordered pair. It follows, then, that the distance between the points whose coordinates are (3,1) and (3,5) is 5 - 1 or 4 (Figure 16).

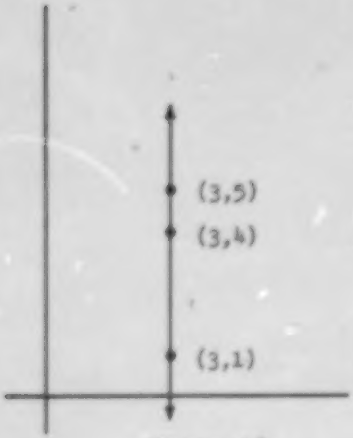


Figure 16

If two points have coordinates such that the first elements of each are different and the second elements are also different, then the line drawn through these points is neither horizontal nor vertical. The ordered pairs (2,3) and (7,4) determine such a line. As we scan this line (Figure 17) from left to right, we notice that it slopes up. We might ask, at this time, if there is any way to compare the "steepness" of the slope of such lines which are neither horizontal nor vertical.

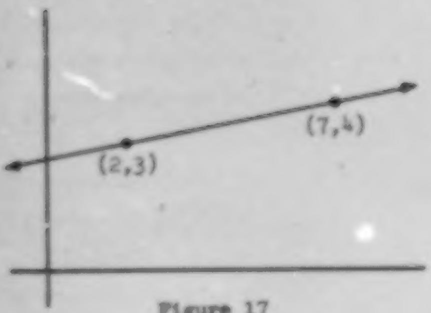


Figure 17

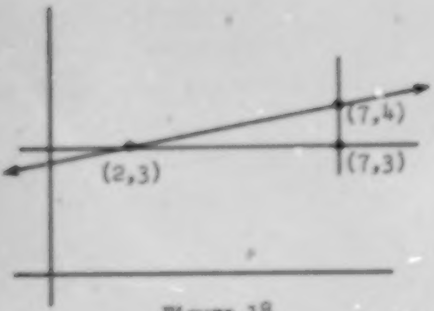


Figure 18

If we draw a horizontal line through the points whose coordinates are (2,3) and a vertical line through the other point, we have two new lines which intersect at a new point. This point is a point of a vertical line which passes through the point (7,4) (Figure 18). By definition of a vertical line, the horizontal coordinate of this new point is 7. This point is also a point of a horizontal line, which, by definition, must have a vertical coordinate of 3. Therefore, the coordinates of this new point are (7,3).

The distance, on the vertical line, between the points (7,4) and (7,3) is 4 - 3 = 1. This vertical distance is often referred to as "rise". The distance, on the horizontal line, between the points (2,3) and (7,3) is 7 - 2 = 5. This horizontal distance is referred to as "run". The ratio of the "rise" to the "run" is called the slope of the line. The slope of the line in this example is $\frac{1}{5}$.

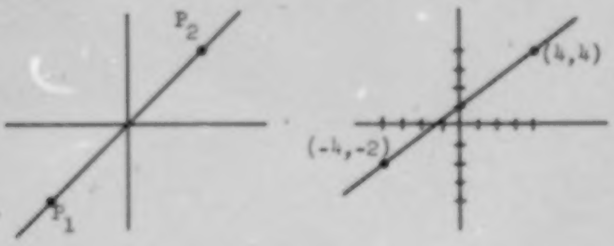
For a straight line the "steepness" is the same all along the line. The slope will be the same between any two points of the line which we might pick. The letter m is usually used for the slope. Thus, for a straight line we have

$$m = \frac{\text{rise}}{\text{run}} = \text{a constant} .$$

We note that in finding the rise we subtracted 3 from 4. These numbers were the second elements of the original ordered pairs which we used to find the line. The run was determined by subtracting 2 from 7. These numbers were the first elements of the original ordered pairs. From this it appears that it is not actually necessary to draw in the horizontal and vertical lines through the points in order to find the slope of the line.

We have defined the slope of a line by using the coordinates of two distinct points on the line. The slope of a given line does not depend on the particular pair of points used to determine the line, nor on the relative position of these two points. The examples below discuss the various possibilities, and show how the value of the slope not only tells the "steepness" of the line but whether it rises or falls as we proceed from left to right. Each of the examples shows a general situation and a specific example.

Example 1: $P_2$ (second point) is above and to the right of $P_1$ (first point).

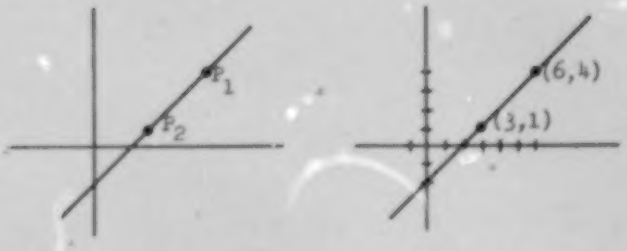


$$m = \frac{4 - (-2)}{4 - (-4)} = \frac{6}{8} = \frac{3}{4}$$

Figure 19

Slope is positive, the line rises as we proceed from left to right.

Example 2: $P_2$ is below and to the left of $P_1$.

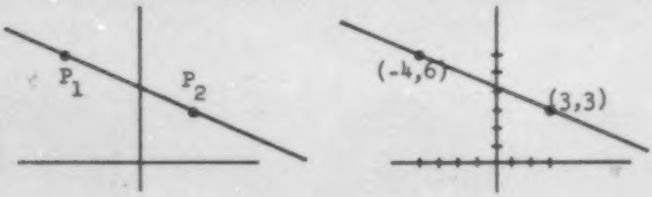


$$m = \frac{1 - 4}{3 - 6} = \frac{-3}{-3} = 1$$

Figure 20

Slope is positive, the line rises as we proceed from left to right.

**Example 3**: $P_2$ is below and to the right of $P_1$.



$$m = \frac{3 - 6}{3 - (-4)}$$
$$= \frac{-3}{7}$$
$$= -\frac{3}{7}$$

Figure 21

Slope is negative and the line "falls" as we proceed from left to right.

**Example 4**: $P_2$ is above and to the left of $P_1$.

$$m = \frac{4 - 0}{-7 - (-4)}$$
$$= \frac{4}{-3}$$
$$= -\frac{4}{3}$$

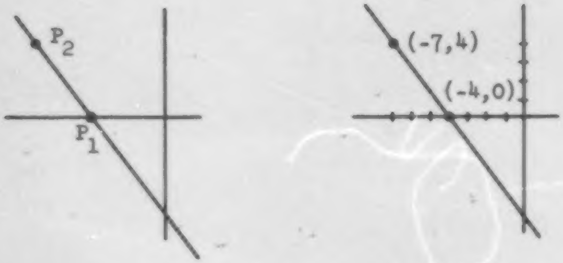


Figure 22

Slope is negative and the line "falls" as we proceed from left to right.

Example 5: $P_1$ and $P_2$ have the same vertical coordinate.

$$m = \frac{3 - 3}{3 - (-4)} = \frac{0}{7} = 0$$

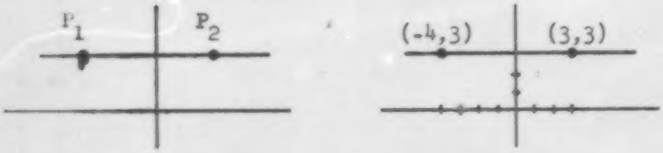


Figure 23

Slope is zero, line is horizontal.

Example 6: $P_1$ and $P_2$ have same horizontal coordinate.

$$m = \frac{-4 - 2}{3 - 3} = \frac{-6}{0}$$

m undefined

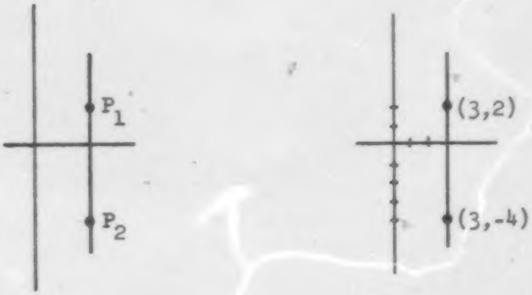


Figure 24

Slope is undefined, line is vertical.

We may summarize the preceding results as follows:

If $m > 0$, the line rises to the right.
If $m < 0$, the line falls to the right.
If $m = 0$, the line is horizontal.
If m is undefined, the line is vertical.

## Exercise 4

1. Which of the following two ordered pairs determine a horizontal line, a vertical line and a line which is neither?

   (a) (3,2), (5,2)
   (b) (0,0), (7,0)
   (c) (10,4), (4,10)
   (d) (5,6), (6,7)
   (e) (2,8), (4,8)
   (f) (2,3), (2,2)
   (g) (561,10), (562,11)
   (h) (3,14), (6,28)
   (i) (9,8), (9,1)
   (j) (0,8), (0,5)

2. For each of the following two ordered pairs, state the rise and the run for the line determined by these points.

   (a) (2,5), (4,8)
   (b) (3,9), (2,1)
   (c) (8.5,7), (9,9)
   (d) (20,10), (25,7)
   (e) (5,3), (5,986)
   (f) (763,763), (25,25)
   (g) (8,7), (2,9)
   (h) (8,10), (0,10)
   (i) (3.7, 12.6), (5.2, 2.1)
   (j) $(\frac{3}{4}, \frac{5}{6})$, $(\frac{5}{4}, \frac{11}{6})$

## 2.6 Absolute Value and Relation

We have already obtained experimentally from our number generator, a relation between S (the numbers on the scale) and T (the number of turns of the indicator). This relation was displayed on coordinate paper as in Figure 25. We found it to be a function. Let us examine the ordered pairs again. Both (1,10) and $(\frac{19}{15}, \frac{38}{3})$ are in the first quadrant, while (-1,10) and $(-\frac{1}{20}, \frac{1}{2})$ are in the second quadrant. We will study these two quadrants separately.

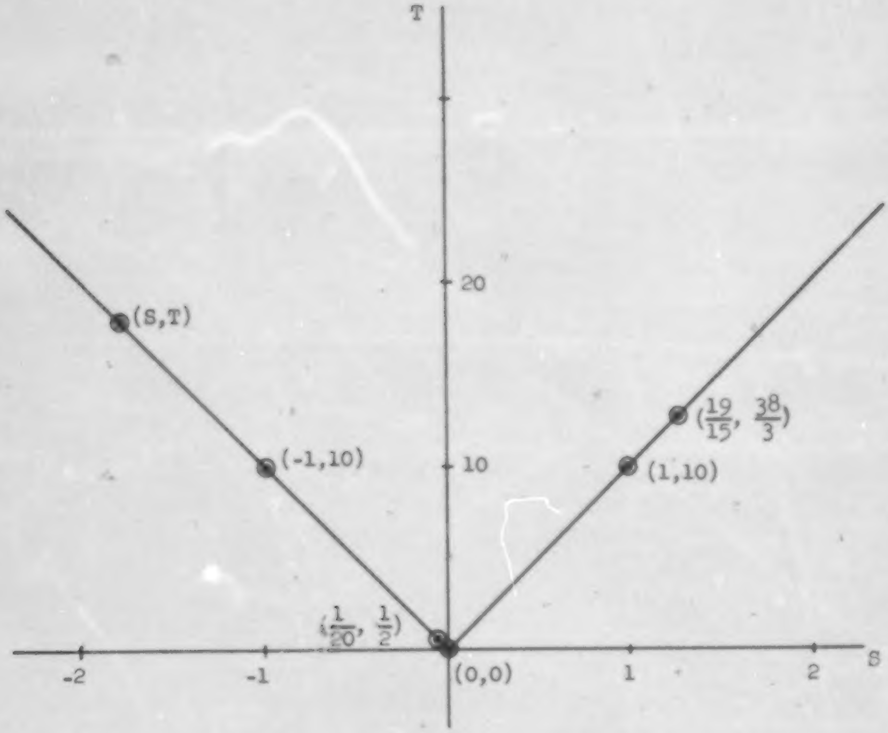
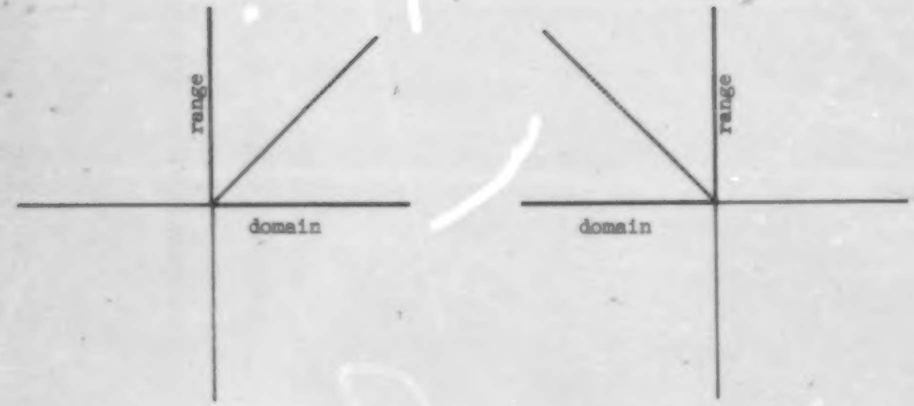


Figure 25

To describe the situation, we must limit our domain. In the first case, our domain will be the non-negative numbers and then we have a linear function in the first quadrant. We call it linear because all points lie on the same straight line (Figure 26a). If we change our domain to the non-positive numbers, we have a linear function in the second quadrant (Figure 26b). Use your answers to Exercise 1 and form (S,T) ordered pairs. Check these ordered pairs to be sure they lie on one of these linear functions. Only the ordered pair (0,0) lies on both.



(a) (b)

Figure 26

The set of ordered pairs in the first quadrant all appear to satisfy the equation

$$T = 10S. \qquad (1)$$

Look, for example, at $(1,10)$ and $(\frac{19}{15}, \frac{38}{3})$ .

We call (1) the linear equation for the graph of Figure 26(a).

The ordered pairs in the second quadrant, for example $(-1,10)$ and $(-\frac{1}{20}, \frac{1}{2})$ satisfy the equation

$$T = -10S . \qquad \text{(See Figure 26(b).)} \qquad (2)$$

In equation (2), the right-hand member, $-10S$, is equivalent to $10(-S)$. The domain of this function, however, is limited to the non-positive real numbers. This means that $-S$ is the negative of a negative number, or a positive number. Recall from Chapter 1, the definition of absolute value,

$$|a| = a \text{ if } a \geq 0$$
$$|a| = -a \text{ if } a < 0 .$$

Another way, then, of saying $-S$, when $S$ is non-positive, would be to say $|S|$. Now our equation can be rewritten

$$T = 10\,|S| . \qquad (3)$$

This form of the equation also applies to the first part of our example where the domain of that part of the function was all non-negative real numbers. Since the absolute value of any non-negative number is that same number, the two equations $T = 10S$ and $T = 10\,|S|$ say exactly the same thing.

The advantage of our discussion is that we now have a single equation over the domain of all real numbers which completely describes the set of ordered pairs in our function. This function is often referred to as an absolute value function.

### Exercise 5

1. Check the ordered pairs you obtained in the scale-turns relation to see if they satisfy either $T = 10S$ or $T = -10S$.

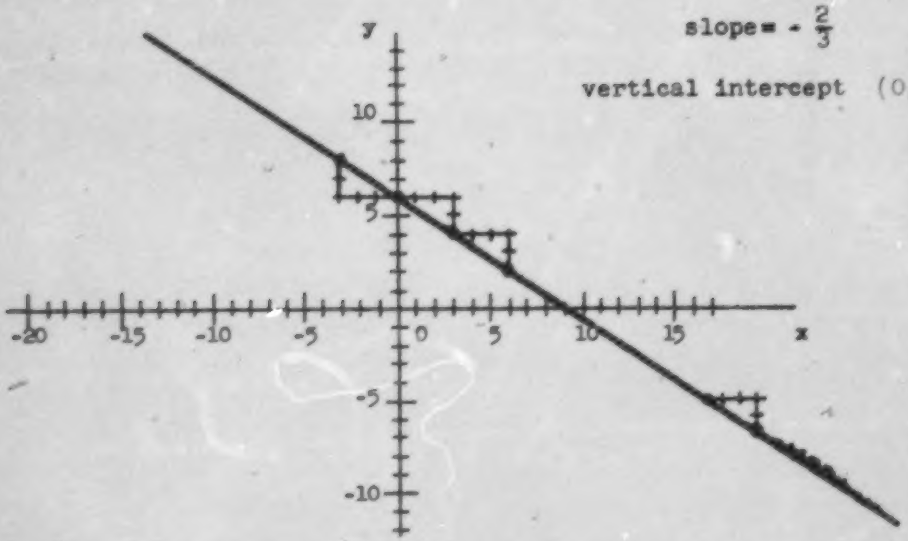
2. Check the ordered pairs you obtained in the faces-scale relation to see if they satisfy either $S = \frac{1}{60} F$ or $S = - \frac{1}{60} F$ .

3. Graph each of the following.

   (a) $y = |x|$ (d) $y = -2\ |x|$
   (b) $y = |-3|x$ (e) $y = 3|-x|$
   (c) $y = 5|x|$ (f) $|y| = x$

## 2.7 Slope-Intercept Form

The vertical intercept of a line is the point on a line where the first element of the ordered pair is zero. This is the ordered pair $(0,b)$. Let us see how the vertical intercept and slope can help us to draw lines. Suppose a line has a vertical intercept $(0,6)$ and its slope is $-\frac{2}{3}$ . Let us draw the line as well as write its equation. To draw the graph, we start at the intercept $(0,6)$. Then we use the slope to locate other points on the line. The fact that the slope is negative tells us that the line will fall as we go to the right, and the number $\frac{2}{3}$ tells us how "fast" the line falls. If we take the point which we know is on the line, $(0,6)$, as one of two points, we can find another point on the line 3 units to the right and 2 units down. We now have two points through which we may draw the line (Figure 27).



slope = $-\frac{2}{3}$

vertical intercept (0,6)

Figure 27

Since the slope is the same for the entire line, any two points of the line will give us the same ratio $-\frac{2}{3}$. Take the vertical intercept (0,6) and another point (x,y) which lies on the line, then the slope $-\frac{2}{3}$ is given by the ratio

$$\frac{y - 6}{x - 0} = -\frac{2}{3}$$

or

$$\frac{y - 6}{x} = -\frac{2}{3} .$$

Multiplying both sides of this expression by x we get

$$x\left(\frac{y - 6}{x}\right) = -\frac{2}{3}x .$$

But $\frac{x}{x}$ is the same as 1, so we can again rewrite to get

$$y - 6 = -\frac{2}{3}x$$

and, finally,

$$y = -\frac{2}{3}x + 6.$$

We can repeat this same process with a line whose vertical intercept is (0,b) and whose slope is m. Let us take any point (x,y) on the line. Since the slope is the same for the entire line, any two points of the line will give us the same ratio m. The slope is given by the ratio

$$\frac{y - b}{x - 0} = m .$$

This equation can be rewritten in the form

$$y = mx + b .$$

It is left to the student to verify that this last equation is correct.

Except a vertical line, every straight line can be given an equation of this form. The equation

$$y = mx + b$$

was derived from our definition of slope and the statement that all portions of the line have the same slope. If we look more carefully at the derivation of this equation, you will recall that we began with two ordered pairs, one of which was of the form (0,b). This point has a special significance. This is a point on the vertical axis. Since we have already said that this line cannot be a vertical line, we know that it can cross the vertical axis at exactly one point whose coordinates are (0,b). This point is referred to as the "y-intercept". Looking again at the equation in this form, we note that the factor m is the slope of the line and the term b gives the intercept. Hence, this form of the equation of a straight line is called the "slope-intercept" form.

Example: Draw the graph of the equation $2x - 3y = 18$.

Solution: This equation is not in the slope-intercept form. However, we can solve the above equation for y.

$$-3y = -2x + 18$$
$$y = \frac{2}{3}x - 6$$

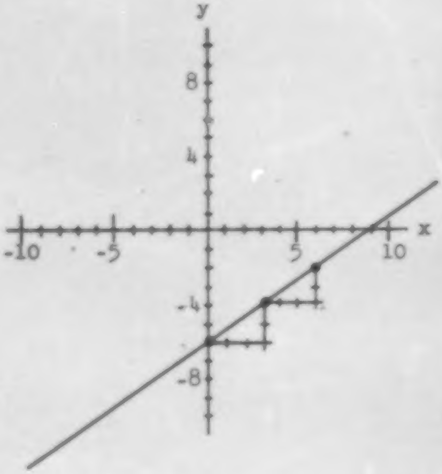


Figure 28

Once we have the equation in this form, we can compare it with $y = mx + b$ and we see that the slope is $\frac{2}{3}$ and the vertical coordinate of the "y-intercept" is -6. Our starting point is therefore the point with coordinates (0,-6). The slope is positive so that line "rises" as we move to the right. The numerical value of the slope $(\frac{2}{3})$ tells us that a horizontal change of 3 is associated with a vertical change of 2. Starting with the point (0,-6) we can easily find another point 3 units to the right and 2 units up (3,-4). Using these two points we may now draw the line.

## Exercise 6

1. Calculate the slopes of lines $\ell_1$, $\ell_2$, $\ell_3$, and $_4$ in Figure 29, using in each case the two points indicated on the lines.

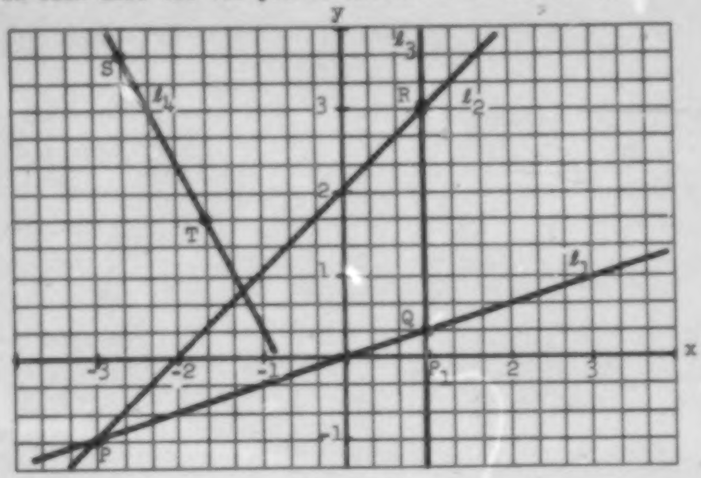


Figure 29

2. What is the slope of a horizontal axis? a vertical axis?

3. With reference to a set of coordinate axes, select the point (-6,-3), and through this point

   (a) draw the line whose slope is $\frac{5}{6}$. What is an equation of this line?

   (b) draw the line through (-6,-3) which has a slope of zero. What is the equation of this line?

4. Draw the following lines:

   (a) a line through the point (-1,5) with slope $\frac{1}{2}$.

   (b) a line through the point (2,1) with slope $-\frac{1}{2}$.

   (c) a line through the point (3,4) with slope 0.

   (d) a line through the point (-3,4) with slope 2.

   (e) a line through the point (-3,-4) with slope undefined. (What type of line has no defined slope?)

5. Consider the line containing the points (1,-1) and (3,3). Is the point (-3,-9) on this line?

   Hint: Determine the slope of the line containing (1,-1) and (3,3); then determine the slope of the line containing (1,-1) and (-3,-9).

6. Write an equation of each of the following lines:

(a) The slope is $\frac{2}{3}$ and the y-intercept number is -2. (The y-intercept number is the vertical coordinate of the point at which the line crosses the vertical axis. In this case the coordinates of the intercept are (0,-2).)

(b) The slope is $\frac{3}{4}$ and the y-intercept number is 0.

(c) The slope is -2 and the y-intercept number is $\frac{4}{3}$.

(c) The slope is -7 and the y-intercept number is -5.

7. What is the slope of the line containing the points (0,0) and (3,4)? What is the y-intercept number? Write the equation of the line.

8. Verify that the slope of the line which contains the points (-3,2) and (3,-4) is -1. If (x,y) is a point on this same line, the slope could be written as

$$m = \frac{y - 2}{x - (-3)} \quad \text{or} \quad \frac{y - (-4)}{x - 3}.$$

Show that both expressions for the slope give the same equation for the line.

9. Write the equations of the lines through the following pairs of points. Use the method of Problem 8.

(a) (0,3) and (-5,2)
(b) (5,8) and (0,-4)
(c) (0,-2) and (-3,-7)
(d) (5,-2) and (0,6)
(e) (-3,3) and (6,0)
(f) (-3,3) and (-5,3)
(g) (-3,3) and (-3,5)
(h) (4,2) and (-3,1)

10. Graph each of the following:

(a) $y = \frac{3}{5}x + 8$
(b) $y = -\frac{8}{3}x - 12$
(c) $3x + 4y = 16$
(d) $y = |x| + 5$
(e) $y = |x - 3|$
(f) $y = 2|x - 1| + 4$

## 2.8 Summary

In this chapter the "real number generator" was used to illustrate the properties of the real numbers.

A *relation* was defined as a set of ordered pairs. The set of all *first elements* in the ordered pairs of the relation is called the *domain* of the relation and the set of all *second elements* is called the *range*. A *function* is a relation such that for each first element in the ordered pairs there is exactly one second element.

The seesaw experiment was used to illustrate the multiplication properties of the real numbers.

$$(+) \cdot (+) = (+)$$
$$(-) \cdot (-) = (+)$$
$$(+) \cdot (-) = (-) \cdot (+) = (-)$$

Slope, m, of a straight line was defined as $\frac{\text{rise}}{\text{run}}$. Slope is *positive* if the line "rises" as we proceed from *left to right*. Slope is *negative* if the line "falls" as we proceed from *left to right*. Slope is zero for a horizontal line. The slope of a vertical line is undefined.

The slope-intercept form of the equation of a straight line,

$$y = mx + b$$

was also developed.

## Chapter 3

# THE FALLING SPHERE

### 3.1 Introduction

In the previous chapter, linear relations were introduced through the Number Generator Experiment. The ordered pairs which were obtained lay on a line because of the way in which they were generated. More often, the data we obtain in an actual experiment will be scattered about somewhat. In this chapter we wish to discuss the way in which such data can be handled. We will find it necessary to introduce the concept of the ideal line which will best fit the data.

Only after this ideal line has been constructed can a mathematical model of the experiment be developed. By looking at the physical systems we can often find new relations which increase our understanding of the structure of the mathematical systems.

### 3.2 The Falling Sphere

This experiment continues our discussion of linear function. You have probably learned in your study of science that all bodies take the same time to fall any given distance in a vacuum. You know, however, that an iron ball and a feather dropped simultaneously from the same height will not reach the floor at the same time. Unless dropped in a vacuum, an object always encounters some form of resistance exerted by the medium through which the object falls. In a fluid medium (a liquid or gas) this resistance is not constant, but increases as the speed of the body increases. Eventually a point is reached at which the upward resistive force equals the downward gravitational pull on the object. From this point on the object will fall at a constant speed.

This steady speed is called the terminal velocity. A man jumping from a plane will reach a terminal velocity of about 120 miles per hour. A "sky diver", with proper control of his body, can lower this figure to about 50 miles per hour. An opened parachute encounters a much greater resistance and lowers one's terminal velocity to a point of relative safety.

To investigate the phenomenon of terminal velocity, a small ball bearing is allowed to fall through a thick fluid (Karo syrup). The ball bearing will reach its terminal velocity in the first few millimeters and then the ball will continue to fall at a steady speed.

As in all experiments, we should think of all the possible variables we are likely to meet, and decide how to handle them. Since our investigation will center around the speed at which the ball falls through the fluid, we must determine which of the variables will influence this speed.

To test the influence of the size of the object upon the terminal velocity, we could drop ball bearings of different sizes into the same container filled with the same liquid.

To test the effect of the jar upon the speed of the falling ball, we could drop the same ball into different size containers filled with the same liquid.

To test the influence of the liquid itself, we could drop ball bearings of the same size into the container filled with different liquids.

If we notice any difference in the terminal velocity of the ball in any of these situations, then the factor that changed is a variable in which we are interested. Can you think of any other variables which may influence the experiment? Does the temperature of the liquid influence the speed of the ball in the same way that it affects the speed of the hot fudge moving off the top of an ice cream sundae?

Once we have our list of variables, we must determine an experimental procedure in which we can control the influence of these variables upon the terminal velocity. We will pick one container and one liquid and always have the ball fall in the same portion of the jar.

The terminal velocity of the ball, however, cannot be measured directly. What we must do is to measure the distance the ball will fall during some time interval. For example, to find the speed of an automobile we have to know the distance traveled and the time taken to travel this distance.

In this experiment we will use a metronome as a timing device, thus providing an audible signal for selected time intervals. In this case we pick the time intervals, and the distances covered by the falling object will then depend on these time intervals. We then have distance as a function of time. In a later experiment we will reverse the roles of time and distance. We will fix the distances and a stop watch will be used to find the corresponding times. Then time becomes a function of distance.

To record the position of the ball as it falls through the syrup, fasten a thin paper tape to the side of the cylinder with cellophane tape.

See Figure 1. Drop a ball bearing into the cylinder so that it falls along the wall of the cylinder as close to one edge of the tape as possible. Since the velocity of the ball will be quite small, only a little practice is needed to follow the path of the ball along the edge of the tape. As the ball moves along the tape, mark its position with a pencil each time you hear the click of the metronome. The metronome should be adjusted to click every two seconds.

A small magnet will be necessary to position the ball along the edge of the tape before releasing it. Hold the magnet against the outside of the glass cylinder and place the steel ball against the inside of the cylinder next to the magnet. The pull of the magnet will hold the ball in place through the glass. When the magnet is pulled away, the ball will begin to fall through the fluid.

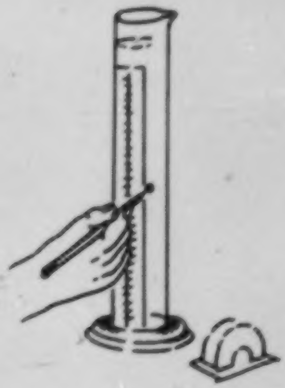


Figure 1

The ball can be brought back up through the fluid by placing the magnet against the outside of the cylinder closest to the ball resting on the bottom of the cylinder. Raise the magnet slowly along the outside of the cylinder. The ball will follow the magnet to the top of the cylinder.

You do not have to mark the path of the ball for its entire fall. Ten position marks taken at two-second intervals will be sufficient for each trial. At least four separate trials of the experiment should be made, using a new tape for each trial. Mark each trial number on the tape and indicate which end of the tape was at the top of the cylinder.

It is not necessary to make the first mark in the same place each time. This first mark is taken to be the position of the ball at "zero" seconds, the second mark, the position at the end of two seconds, etc.

### 3.3 Tabulating Data

After completing the four trials, fasten each tape in turn to a centimeter ruler so that the "zero" time coincides with one of the ruler marks.

**Measure the distance in millimeters from the "zero" mark to the first mark, from the "zero" mark to the second, etc. (See Figure 2.)**

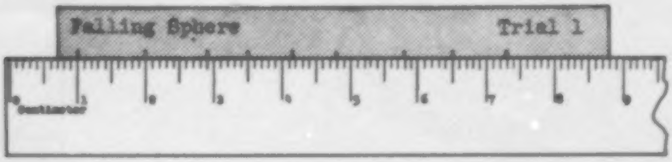


Figure 2

**Record the data for all four trials in a table of the form shown in Table 1.**

**THE FALLING SPHERE EXPERIMENT**

| Time t (seconds) | Trial 1 Distance d (millimeters) | Trial 2 Distance d (mm) | Trial 3 Distance d (mm) | Trial 4 Distance d (mm) | Average Distance (mm) Guess | Average Distance (mm) Calc. |
|---|---|---|---|---|---|---|
| | | | | | | |

Table 1

### 3.4 Analysis of Data

If the data is examined, we see that our table associates a certain value for the distance the ball has fallen (d) with a certain value of the time (t). The table shows that there is a relationship between the time and the distance the ball has fallen. The value we obtain for the distance depends on the time and, therefore, our data forms a set of ordered pairs. As we have seen before, we can represent ordered pairs of numbers as points on the coordinate plane. In doing the experiment we have decided what time intervals to use, and the resulting distance the ball has fallen depended on this time interval. The general practice is to make the value of the variable that we controlled the first element in the ordered pair. Thus, for this experiment, the first element in the ordered pairs will be the time value, and the second element will be the distance associated with this time value. Our ordered pairs become (t,d) pairs. It will be helpful to label the horizontal axis the time axis and the vertical axis the distance axis.

We are going to use the graph of these ordered pairs for an analysis of the behavior of the falling sphere, and it will be advantageous to have the graph "fill" the paper as much as possible. We know that all of our points will fall in the first quadrant, because all of our values for time and distance are positive. Instead of drawing the axes, as in Figure 3(a), we use the form suggested in Figure 3(b).

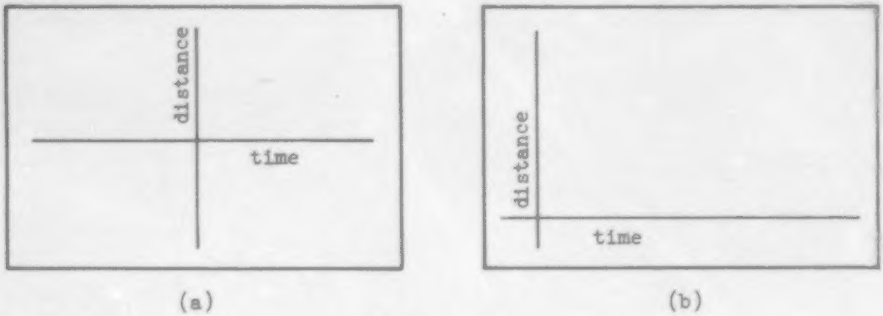


Figure 3

Now our data will not be crowded into one corner of the graph paper and we can make finer divisions along the axes. The distance scale should be in millimeters, and the time scale in seconds. Once the data is plotted, you probably will have a graph which looks something like that in Figure 4.

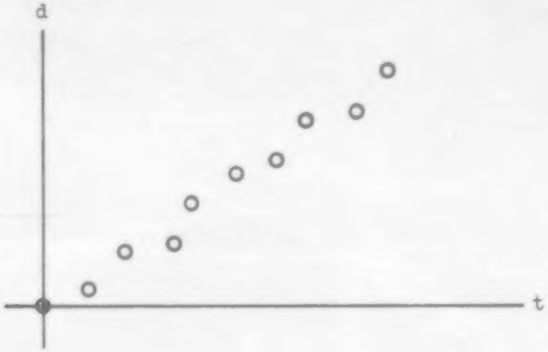


Figure 4

Once the scales have been set and the data plotted, we have the problem of interpreting the meaning of the space between these points. In terms of the physical setting, we can argue that the space between adjacent points should be "filled in". If we had decreased the time interval to one-tenth of a second each time, instead of by two seconds, we would have found a new distance reading for each time. Even though reading the change in position for such a small time change would be difficult, the sphere would make a very small change in its position for every time change. We now have to decide how the position of the sphere would change with time. The variation is probably quite regular with two-second intervals. There is no reason to suppose that a regular variation of position would not occur between these points. Our first guess as to a model of the behavior of the sphere with respect to time would then be to join our experimental points with straight line segments. This procedure will give us something like the graph shown in Figure 5.

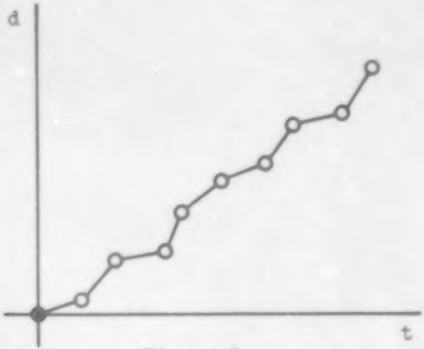


Figure 5

This method of joining our experimental points is perhaps not the most accurage model we can construct. When we say that the sphere behaves exactly like our experimental points, we are saying that our readings are exact. Can you think of any reasons for your data not being exact? This graph is also the result of a single trial of the experiment. Scientists and mathematicians do not like to generalize the results of a single trial. The errors of measurement may be great enough to make the model obtained not very meaningful.

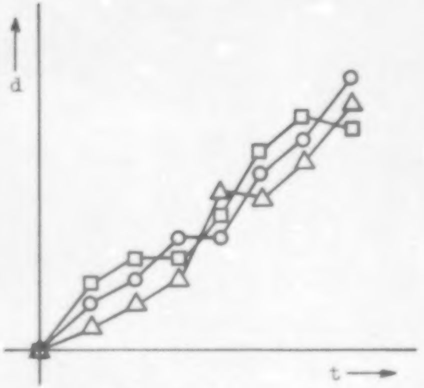


Figure 6

If we were to repeat the experiment a number of times and graph the data on the same coordinate plane, you would probably arrive at a figure like that shown in Figure 6. This figure shows us something about our ability to reproduce the experiment. Do we obtain about the same ordered pairs a second and third time? It also suggests that the "spread" of the plotted points may be due to certain inaccuracies involved in the measurements. Perhaps the plotted "points" should not be points at all, but small areas. We are led to the conclusion that the results of an actual experiment, as contrasted to those of an ideal experiment with perfect equipment and exact measurements, are two different things. We have at this point a relationship between distance and time in the form of a data table and in the form of a graph. What we desire now is the graph which will explain the ideal behavior of the sphere. The data from each trial, and the *braid* arrangement of the data seem to suggest a straight line. You probably cannot find a straight line which will connect all the points for any one trial. However, with a little practice, you should be able to find a line which seems to "best" represent all of the data. This "best straight line" will be our *physical model* of a relation we have "guessed". This line represents our model of an ideal experiment. (See Figure 7.)

Once we have decided to depart from the experimental "facts" and draw a single straight line to represent our data, we have a graph similar to that in Figure 7. This graph gives a pictorial relation of time and distance. Our problem now is to find a mathematical representation of this relation. We now have a relation between time and distance in terms of tabular data and a graph of this data. We have also formed a physical model to represent an idealized version of this data. We now want to obtain a mathematical model which will describe the position of the sphere in terms of the time. This is our third step in the analysis of the experiment.

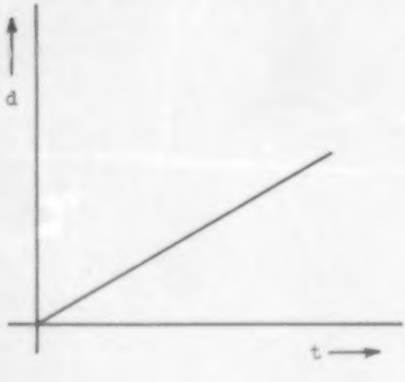


Figure 7

## 3.5 Graphing the Experimental Data

The data you recorded for all four trials should be graphed on a single sheet of coordinate paper.

If you graph the "braid" arrangement discussed above, all of the points should fall in some fairly narrow band. (See Figure 8.) Do you think that if you were to repeat the experiment under the same conditions that your new points would fall within this band?

We obtain a band rather than a line because of the various errors in measurement and the influence of variables other than distance and time. An analysis of the effects of these will be reserved for a future course.

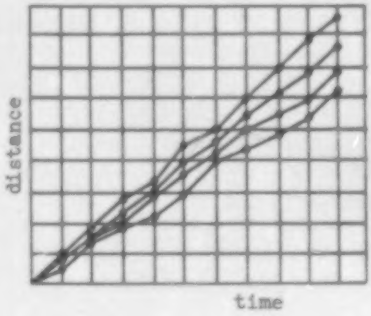


Figure 8

There are many straight lines we could select to represent an idealized relationship between time of fall and distance. No single straight line will connect all of the points for any trial. With thought and care use a ruler to draw a line which you think best represents all the data from all your trials. Your "best straight line" represents the model of an ideal experiment and becomes the physical model of the relation.

Remember to include the (0,0) point in your line. The manner in which we performed the experiment tells us that at "zero" time the ball has fallen "zero" distance. Thus, even though there are many lines to choose from, every one of them should pass through the origin.

We still have to build a mathematical model of the physical relationship shown in our "time-distance" graph. We can do this by repeating the procedure learned in the Number Generator Experiment. The slope should not be difficult to compute, for we know that the line must pass through the origin; hence, the coordinates of the "y" intercept are (0,0). The equation which describes the motion of the falling sphere is therefore quite simple. Calculate the slope, using any two points on the line. Since the coordinates of the origin are (0,0), this would be a convenient first point to use.

Choose any arbitrary second point on your line with coordinates (T,D). Following the procedure used in Chapter 2, we have

$$\frac{D - 0}{T - 0} = m .$$

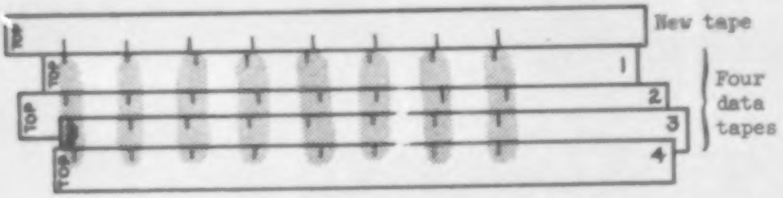
Using this value for the slope, the equation relating time and distance in the experiment becomes

$$d = mt .$$

The slope in this experiment has a special significance. The vertical distance from the first point to the second is a number of millimeters, while the horizontal difference between these points is a number of seconds. The slope, defined as the ratio of these two differences, will be expressed in units of millimeters/second. The value of the slope is defined as the measure of the velocity of the ball. Since we have found that the experiment yields a straight line, the slope and, therefore, the velocity, is a constant. Our initial comments are thus confirmed--by the time we begin taking data, the ball has already reached its terminal velocity and now falls at a constant rate.

### Exercise 1

1. Reproduce the "best straight line" you have drawn to represent the data of this experiment on a clean sheet of coordinate paper. Take the four pieces of paper tape used to mark the position of the ball and arrange them so that the zero marks are in line. On a clean fifth tape make a mark to indicate a "zero" position and align this mark with the other zero marks. The other marks on your tapes will not be "in line", but should tend to center in groups. Make a mark on the clean tape to indicate your "guess" as to the position which best represents each vertical set of marks. Using the fifth tape as if it were a new trial, mark your measurements in the usual way, enter the data in your table, and graph the ordered pairs. Do these points come closer to forming a straight line than any of your four trial runs? How does this line compare with the "best line" you drew from the "braid" arrangement?



2. From the data of your four trials, find the average distance traveled by the ball in each time interval. To do this, add the distance in each row of the trials in Table 1 and divide by the number of trials. Make a new column in your table, "Average Distance (mm)", and now plot average distance versus time on the same sheet of coordinate paper used for Problem 1. How close do these points come to forming a straight line? You now have three lines on this coordinate plane. The first is the "best straight line" from your original data, the second is the line obtained in Problem 1, and the third line is the one obtained by the process of averaging. How do these three lines compare?

3. Draw the 1st quadrant using a scale of 1 second for each horizontal division and 1 millimeter for each vertical division. Draw a line which passes through the origin and has a slope of 1 mm/sec; 2 mm/sec; and 3 mm/sec. Label these lines $\ell_1$, $\ell_2$ and $\ell_3$.

4. Plot the lines in the preceding problem, using a horizontal scale of 1 second per division, and a vertical scale of 0.5 millimeter per division. Compare these three lines with the lines in Problem 3.

5. Draw a 4th quadrant on a sheet of coordinate paper. Use the same horizontal scale (in seconds) that you used to represent the data from the Falling Sphere Experiment. Make a negative distance scale (in millimeters) along the vertical axis. Note that this was the orientation of your scale when you performed the experiment. Plot the time-distance data from your experiment on this sheet and draw the "best" line. Calculate the slope. What is the significance, if any, of a negative velocity?

## 3.6 The Point-Slope Form

When we plot data obtained from different experiments involving linear relations, we always obtain a "best" straight line. The orientation of this line on the coordinate plane will vary from experiment to experiment. We can, however, discuss three general types. In Chapter 2 we found that a line which intersects the vertical axis at a point other than the origin would have an equation of the form $y = mx + b$, as illustrated in 9(a). In the Number Generator Experiment and the Falling Sphere Experiment, the graphical representation of the data passed through the origin, Figure 9(b). We found that all graphs of this type could be represented by an equation of the form $y = mx$.

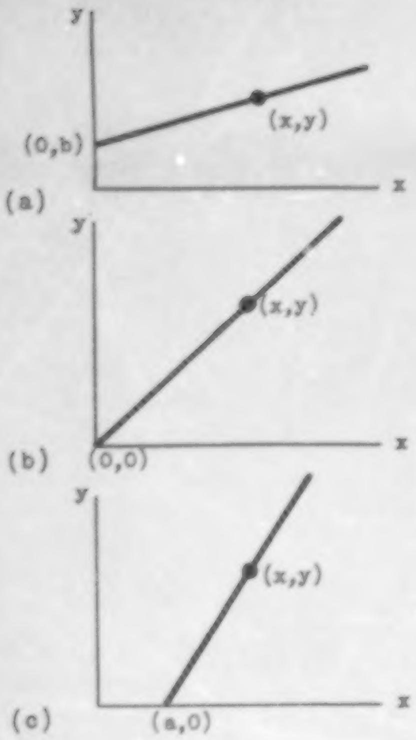


Figure 9

Suppose, however, we are to arrive at a graph which looks like that in Figure 9(c). In this case, if our domain is limited to values greater than a, we will not have a "y-intercept at all". The slope, however, can still be calculated in the usual way by selecting any two points on the graph and finding the ratio of the vertical distance between these points to the horizontal distance between them. The slope is the same for any two points on a straight line. To obtain the equation of this line, the point at which the line intersects the horizontal axis is taken as our first point, and this point has the coordinates (a,0). Then for any arbitrary point with coordinates (X,Y) we can find the slope at this point.

$$\frac{Y - 0}{X - a} = m .$$

Using this value of the slope, the equation relating x and y can be written

$$y = m(x - a) .$$

This is the third of three "special" forms of the equation of a straight line. It is not necessary to remember all three forms. Instead, we can find a general representation for every straight line by using the slope of that line and any point on the line. In Figure 10 we have a point whose coordinates (c,d) are known. If we have previously calculated the slope (m) of this line, then, for any arbitrary point (x,y) we have

$$\frac{y - d}{x - c} = m$$

from the definition of slope, and thus

$$y - d = m(x - c).$$

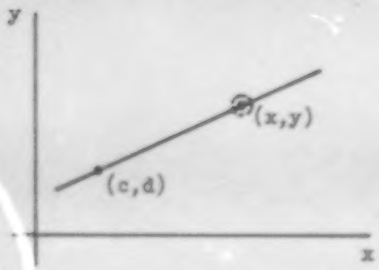


Figure 10

This more general form of a linear equation is called the "point-slope" form. This form of the equation of a straight line will yield each of the three special forms simply by selecting the appropriate special point in each instance. This is done below.

If the graph intersects the y-axis, the coordinates of the point of intersection are (0,b). These values inserted in the point-slope equation gives

$$y - b = m(x - 0) ,$$

and then

$$y = mx + b \text{ (the "slope-intercept" form).}$$

If the graph happens to pass through the origin, we can make use of the coordinates of this point, (0,0), and obtain

$$y - 0 = m(x - 0)$$

and

$$y = mx .$$

In a similar manner, if the line intersects the x-axis, the point of intersection has coordinates (a,0) and we obtain

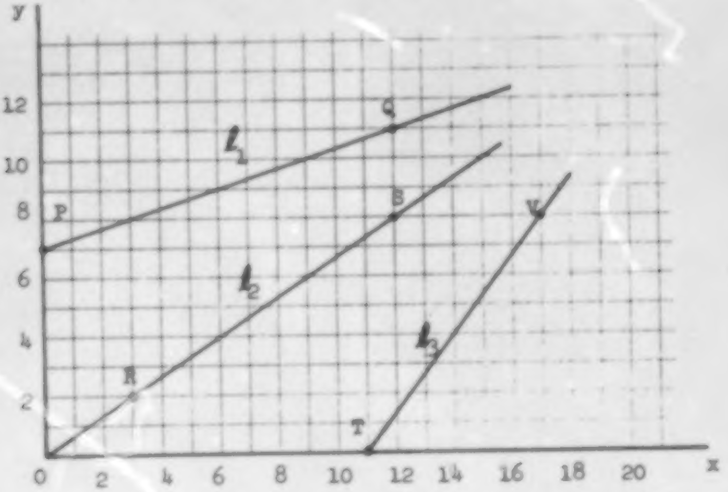
$$y - 0 = m(x - a)$$

or

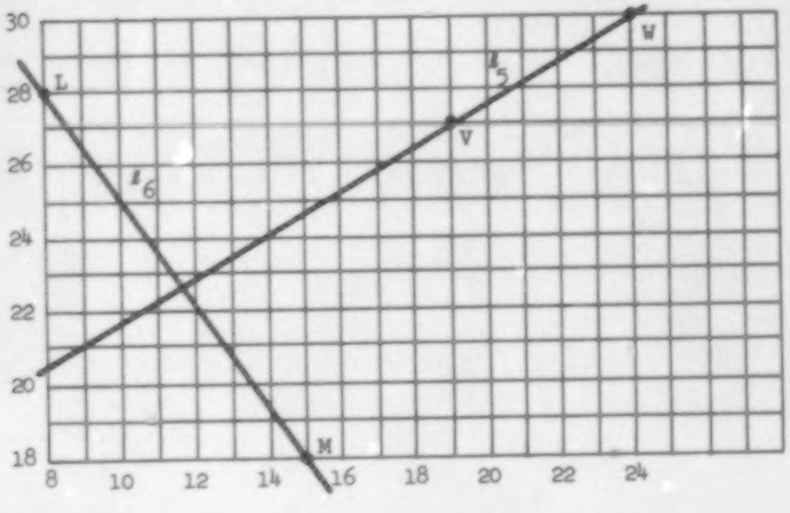
$$y = m(x - a).$$

## Exercise 2

1. Write the equations of the lines $\ell_1$, $\ell_2$ and $\ell_3$, using the two points indicated in each case.



2. Write the equations of the lines $\ell_5$ and $\ell_6$.



3. Find the x and y intercepts for lines $\ell_5$ and $\ell_6$.

Do not extend the lines to obtain a graphical solution. Remember that the y-intercept is the point for which $x = 0$, and the x-intercept is the point where $y = 0$.

4. Refer to your time-distance graph obtained in the Falling Sphere Experiment. Using a point not on the vertical axis together with the slope, find the equation to represent the best straight line. Show that this is equivalent to the equation obtained using the slope-intercept form.

5. The following equations are expressed in point-slope form:

$$y - 6 = 3(x + 4)$$

$$y + 2 = -2(x - 3)$$

$$y + 7 = \frac{5}{2}(x - 2)$$

$$y - 0.5 = -4(x + 3.5)$$

Solve each of these for y. State the slope of the line and the y-intercept in each case.

6. Take your graph of the data obtained in the Loaded Beam Experiment, fit a "best" line and obtain an equation of this line, using the slope-intercept form and the point-slope form.

### 3.7 Relations and Converses

A graphical representation is perhaps the most illuminating way to present a relation. It conveys at a glance much important information. For example, in Figure 11(a) a graph of a semi-circle of radius five is shown. The graph intersects the horizontal axis at two points, (-5,0) and (5,0) and the vertical axis at the point (0,5). Figure 11(b) is labeled to indicate the domain and the range. The domain is the set of numbers (d) such that $-5 \leq d \leq 5$. The range is the set of numbers (r) such that $0 \leq r \leq 5$. Figure 11(c) shows a line segment in the first quadrant. What are the domain and range of this relation?

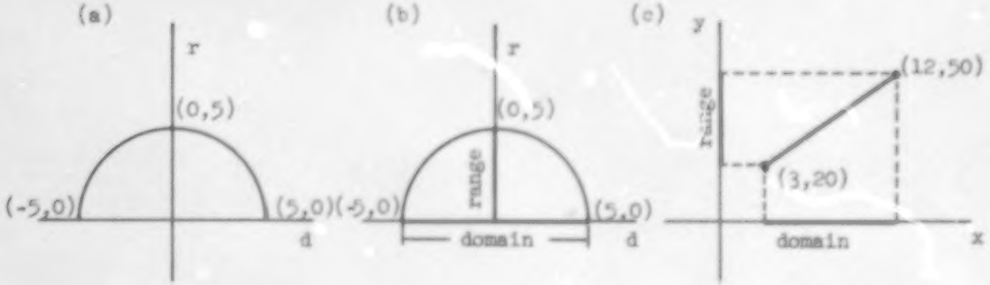


Figure 11

In the Falling Sphere Experiment we have a relation between the ordered pairs which is of the form (time value, distance value). If we take this set of ordered pairs and interchange the first and second elements in each pair, we will obtain the converse of the relation. This means that we will have ordered pairs of the form (distance value, time value). The set of distance values will now form the domain and will be plotted along the horizontal axis, and the set of time values will become the range and be plotted along the vertical axis.

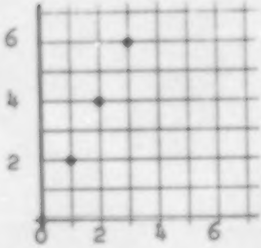
The example below gives a set of ordered pairs A and information about the relation and its converse.

Example:

$$A = \{(0,0), (1,2), (2,4), (3,6)\}$$

$$(\text{converse}) = \{(0,0), (2,1), (4,2), (6,3)\}$$

Graph of A

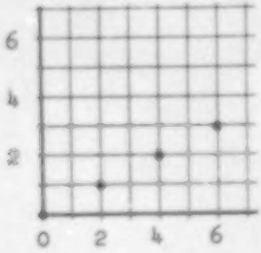


domain {0, 1, 2, 3}

range {0, 2, 4, 6}

relation is a function

Graph of Converse



domain {0, 2, 4, 6}

range {0, 1, 2, 3}

converse is a function

## Exercise 3

(a) Graph the ordered pairs given below, state the domain and range and tell if the relation is a function.

(b) In each case form the converse relation by interchanging the first and second elements of the ordered pairs. Graph the converse, state the new domain and range and tell if the converse is a function.

1. Q = {(2,3), (2,4), (2,5)}
2. M = {(5,3), (6,3), (7,3)}
3. N = {(3,6), (3,-2), (4,-2)}
4. P = {(-1,-3), (-2,-5), (-3,-7)}

## 3.8 Inverse Functions

In Problem 4 above you should have reported that both the relation and its converse met the definition of a function. When this situation occurs, we say that the relation and its converse are inverse functions. With this definition we see that every relation will have a converse but not every function has an inverse. From this point on, when we refer to an inverse we mean that the relation in question is a function and that its converse is also a function.

We can use the graph of a relation to tell if the relation is a function, as described in Chapter 2. The graph can also be used to tell us if the function has an inverse, or, in other words, to tell us if the converse of any relation is a function.

Recall from Chapter 2 that if no line parallel to the vertical axis meets the graph of a relation in more than one point, then the relation is a function. It is not necessary to draw the converse relation to decide if the function has an inverse. If no line parallel to the horizontal axis meets the graph of relation in more than one point, then the converse of the relation is a function. By a combination of these two graphical tests we can decide if a function has an inverse. (See Figure 12.)

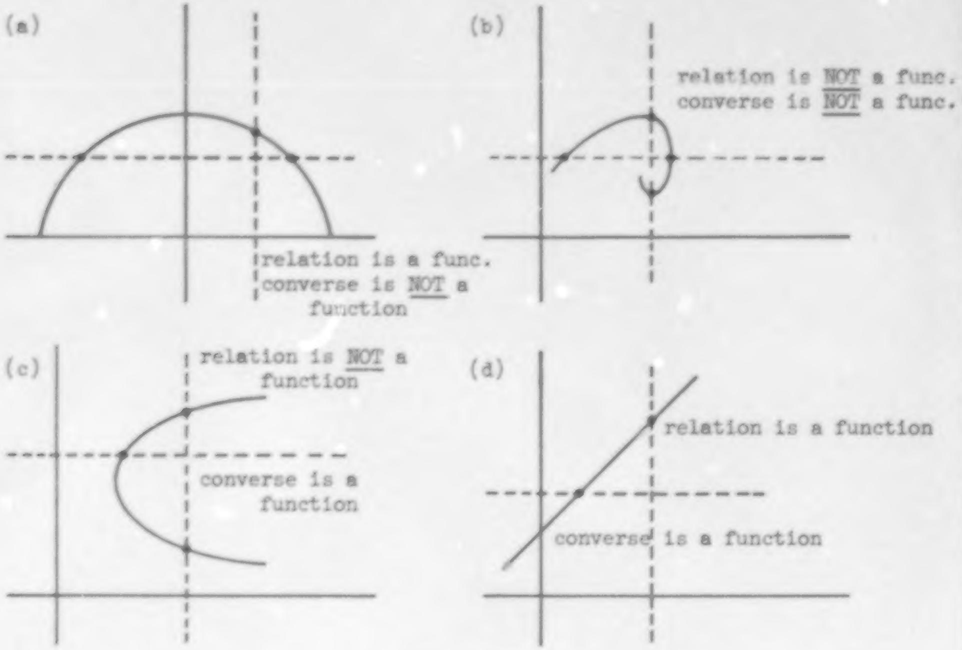

(a)
relation is a func.
converse is NOT a function
(b)
relation is NOT a func.
converse is NOT a func.
(c)
relation is NOT a function
converse is a function
(d)
relation is a function
converse is a function

Figure 12

Just as relations which meet certain qualifications are put in a special class called functions, functions which have an inverse are given a special name. They are called one-to-one functions. Every element in the domain will yield one element in the range and every element in the range will yield one element in the domain.

## Exercise 4

1. Refer to Exercise 2 in Chapter 2. For each of the graphs, check to see if the converse of the relation shown is a function. Are any of these relations one-to-one functions?

2. In the Falling Sphere Experiment, the data in the table forms a relation.

   (a) What are the domain and range of this relation?

   (b) Is this relation a function?

3. Does the "best straight line" describe a function?

4. Are the domain and range of the "best straight line" relation the same as the domain and range of the "data relation"? Explain.

5. Are the domain and range of the equation the same as the domain and range of the graph of the best straight line?

6. In the Falling Sphere Experiment we obtained the equation $d = mt$. Obtain the converse relation by algebraic means. (Hint: solve the equation for $t$ in terms of $d$.) How might we have conducted the experiment to give the converse relation directly?

7. Do the original Falling Sphere relation and its converse form one-to-one functions?

## 3.9 Graphical Translation of Coordinate Axes

A line drawn on coordinate paper always represents some sort of linear function. In Section 3.6 we learned that we can write the equation of a line if we know its slope and the coordinates of one point on the line. In general, if the slope of the line is $m$ and the coordinates of one point on the line are $(c,d)$, the equation of the line is

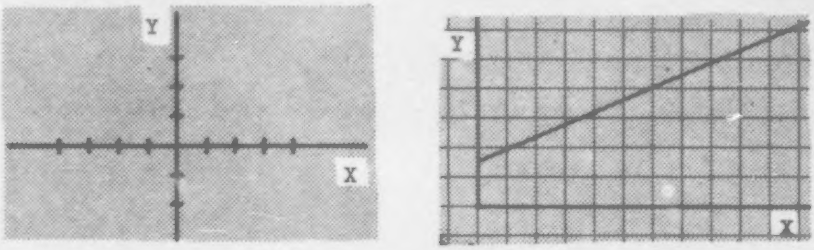
$$y - d = m(x - c) .$$

This general form of a linear equation is called the "point-slope" form. The constants $c$, $d$ and $m$ in this equation determine the location and orientation of the line on the coordinate plane.

For many purposes it is very useful to think of all lines that can be drawn as different positions of a single line. It is only the mathematical description of the line that differs. One point of view would be to think of the line as having moved from one position to another with respect to the coordinate axes. It is also possible to think of the coordinate axes as having shifted with respect to the line. This latter approach is the one we will discuss in this section.

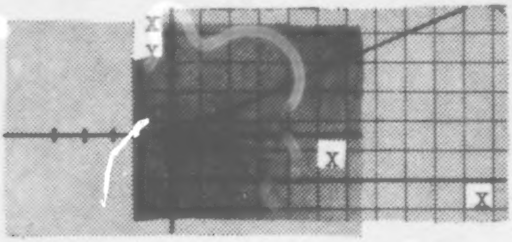
An excellent way to visualize this translation is to have the coordinate axes drawn on a transparent sheet which can then be moved about over the figure. An $8\frac{1}{2} \times 11$-inch sheet of frosted acetate provides a good surface upon which a set of movable coordinate axes may be drawn. In making the overlay, the frosted side of the acetate should be up. Pencil lines can easily be drawn and erased on this surface. The "moving" axes must have the same scale as those on the coordinate axes which are to be translated.

When the plastic sheet is placed upon a regular sheet of graph paper, the graph beneath is easily visible. In this way the graph can be readily related to the "new" coordinate axes carried by the overlying plastic sheet. The new axes may be placed in any position you wish. The sheet of frosted acetate, a piece of graph paper, and the combination of the two are shown in Figure 13(a), (b) and (c).



Acetate sheet with axes.
(a)

Coordinate paper and graph.
(b)



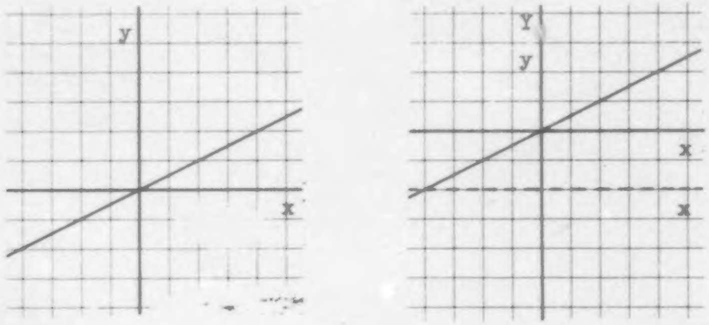
Graph viewed in relation to new axes X and Y
(c)
Figure 13

Figure 13(c) shows the coordinate axes X and Y displaced upward with respect to the origin of the graph beneath. The use of the capital letters X and Y on the overlay will help us to remember that these represent the axes that are translated.

It must be realized that if we are to allow any kind of motion of the coordinate axes X and Y, this motion might be rather complicated. We can simplify matters, however, by recognizing that any complex motion may be broken into two parts. One of these parts is simple straight line motion, called a translation, and the second is rotation. Any motion of the coordinate axes is given by a combination of these two types. Only straight line motion of the axes will be considered here. There is one other important point to be made. Any motion of translation can be considered as made up of two translations, one in the horizontal direction and one in the vertical direction.

Suppose we start with the X and Y axes on the plastic overlay coincident with the x and y axes on the sheet underneath. When these axes are translated, the entire plastic sheet moves horizontally and vertically and is not rotated. The X axis must always remain parallel to the original x axis and the Y axis must always remain parallel to the original y axis.

Figure 14(a) shows the graph of a linear function and Figure 14(b) suggests one of the many ways in which the coordinate axes may be shifted. The axes have been moved upward until the new origin is at the original y-intercept Using this new position of the axes, the equation of the line would now be of the form $Y - mX$, where before it was of the form $y - d = m(x - c)$.



(a) (b)

Figure 14

Notice that the slope of a line never changes as the axes are translated. This is an extremely important feature of a linear translation. When the axes are rotated, this statement is no longer valid.

Example:

Suppose, as in Figure 15, we have a line which passes through the origin. The slope of this line is $\frac{3}{2}$, and the equation of the line is $y = (\frac{3}{2})\,x$.

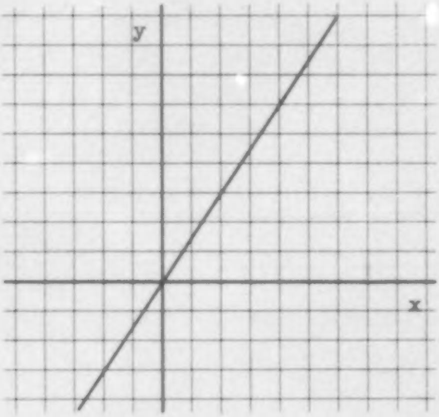


Figure 15

Let us now translate the coordinate axes two units to the left and four units downward. This new situation is shown in Figure 16. The shifted axes are labeled, as before, X and Y, and the original axes are shown as dotted lines.

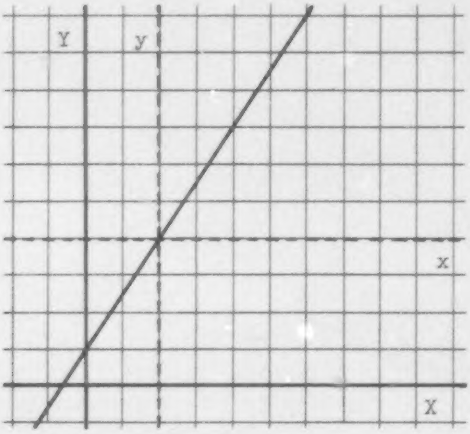


Figure 16

You should now verify that the slope of the line is still $\frac{3}{2}$ . The coordinates of the old origin are now (2,4), and the equation of the line is now $Y - 4 = (\frac{3}{2})(X - 2)$ . This new equation is derived from the point-slope form.

## Exercise 5

1. With reference to a set of coordinate axes, select the point (2,3) and through this point draw the line whose slope is $\frac{1}{2}$ . What is the equation of this line? Use your plastic overlay to obtain the new equation of this line when the origin is shifted:
   - (a) to the left 3 units;
   - (b) to the right 3 units;
   - (c) 4 units upwards;
   - (d) 4 units downwards;
   - (e) to the left 3 units and up 4 units;
   - (f) to the left 3 units and down 4 units.
2. With reference to a set of coordinate axes, draw the line which passes through the points (1,7) and (7,5). What is the equation of this line? Use your plastic overlay to obtain the new equation of this line when the origin is shifted:
   - (a) to the x-intercept;
   - (b) to the y-intercept;
   - (c) to the point (4,6) .

## 3.10 Algebraic Translation of Coordinate Axes

The mathematical description of a graph may be obtained easily by using the graphical procedure described in the preceding section. It is also desirable to be able to describe a graph after the axes have been translated without resorting to the analysis of the graph itself.

First we will show that the point-slope representation of a line can be considered as one in which the coordinate axes have already been translated in both horizontal and vertical directions.

Suppose we have a line which passes through the origin (Figure 17). The equation of this line is $y = mx$ .

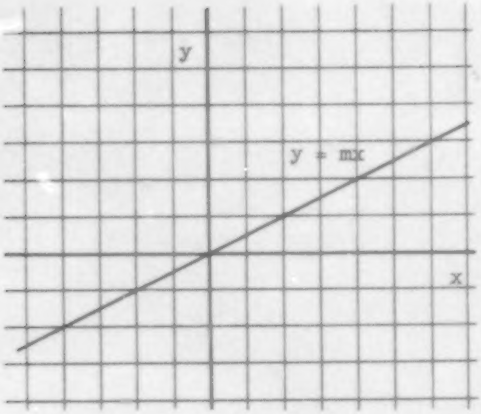


Figure 17

Let us now translate the coordinate axes "c" units to the left and "d" units downward. The shifted axes are labeled, as before, X and Y (Figure 18).

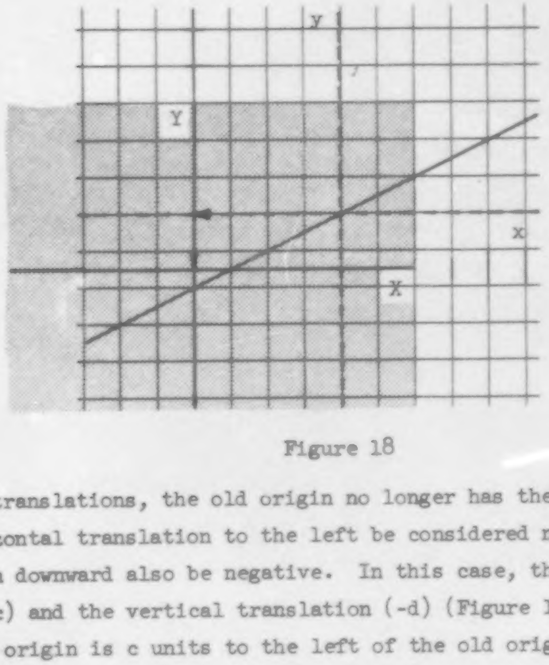


Figure 18

After the translations, the old origin no longer has the coordinates (0,0). Let a horizontal translation to the left be considered negative and a vertical translation downward also be negative. In this case, the horizontal translation is (-c) and the vertical translation (-d) (Figure 19). The position of the new origin is c units to the left of the old origin. Therefore, the new horizontal coordinate of the old origin is c. Similarly, the new vertical coordinate is d.

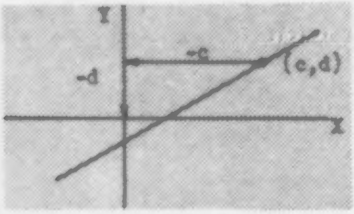


Figure 19

Since the point (c,d), the coordinates of the old origin, is a particular point on the line, we can now describe the line in the familiar point-slope form as

$$Y - d = m(X - c) \ .$$

If we now write this same expression in slightly different form

$$Y + (-d) = m \ [X + (-c)],$$

we may draw an interesting conclusion. Since the quantities in parentheses are the horizontal and vertical translation distances, this last equations tells us that the point-slope representation of a line is given by setting the Y-coordinate plus the vertical translation equal to the slope of the line times the quantity, X-coordinate plus the horizontal translation.

$$Y + (\text{vertical translation} = m \ X +[(\text{horizontal translation})]$$

The procedure described above is a general one, even though it was derived for the particular case of a line passing through the origin. Suppose, for example, we have the line shown in Figure 20. The equation of the line is

$$y - 4 = \frac{1}{2}(x - 4) \ .$$

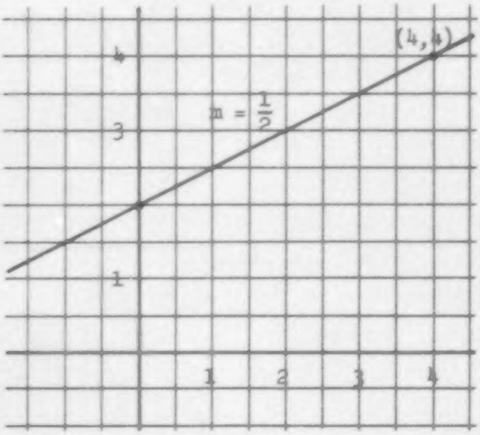


Figure 20

Let us translate the axes, this time with a horizontal translation of two units to the right and a vertical translation of one unit up. This translation is shown in Figure 21. The original axes are shown with dotted lines. In relation to our new axes, every point on the line has a new pair of coordinates (x-2,y-2). The slope of the line has, of course, not changed.

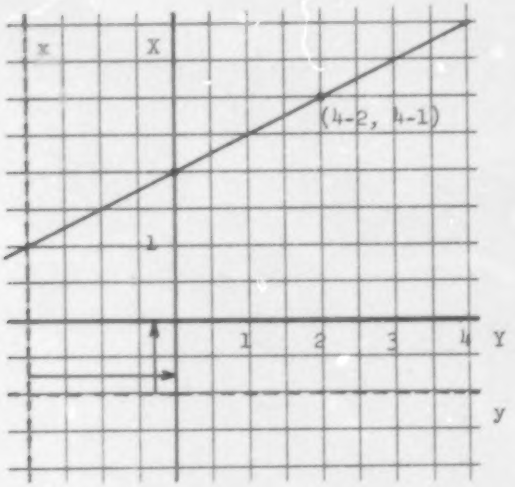


Figure 21

Let us now use the point-slope method to find the equation of this line. The point we used originally had coordinates (4,4); with respect to the new axes its coordinates are now (4-2,4-1). Thus, the equation of the line is now

$$Y - (4-1) = \frac{1}{2}\,[X - (4-2)]$$

and again rewriting in a slightly different form,

$$Y - 4 + (2) = \frac{1}{2}\,[X - 4 + (2)].$$

Simplified, the new equation of the line becomes $Y - 3 = \frac{1}{2}X - 1$, or $Y = \frac{1}{2}X + 2$ with respect to the new origin.

We normally designate a horizontal translation by the symbol h and the vertical translation by the symbol k. As previously stated, a horizontal translation to the right is positive (left is negative) and a vertical translation upward is positive (downward is negative). We can now make a general equation to represent the mathematical description of a line which results from a translation of axes from any previous point. If the original description of the line was

$$y - d = m(x - c)\ ,$$

a horizontal translation of h units and a vertical translation of k gives a new expression

$$Y - d + k = m [X - c + k].$$

## Exercise 6

1. With reference to a set of coordinate axes, draw the line which passes through the points (4,8) and (0,0). What is the equation of this line? Obtain the equation of this line algebraically when the origin has been translated:

   (a) to the left 3 units;
   (b) to the right 4 units;
   (c) to the left 3 units and down 4 units.

2. With reference to a set of coordinate axes, draw the line which passes through the point (1,7) and (7,5). Write the equation of this line in point-slope form. Obtain the equation of this line algebraically when the origin has been translated:

   (a) to the x-intercept;
   (b) to the y-intercept;
   (c) to the point (4,6).

   Compare your results to those obtained graphically in Problem 2 in Exercise 5.

## 3.11 Summary

Using the Falling Sphere Experiment to provide the data, we investigated the phenomenon of terminal velocity. From this data it was also found that a "best straight line" could be drawn which is an idealized representation of the data. This idealized line is a physical model of the relation. It then followed that from the physical model it was possible to develop a mathematical model of the data.

The slope-intercept form of the linear equation was derived to assist us in work with linear functions. It also followed that relations, their converses, and inverse functions could be readily developed. The one-to-one functions were then introduced.

Finally, the translation of axes was investigated. Two separate procedures were used. First, the translation was performed as a physical process using a piece of frosted acetate to clarify the meaning of the translation. Next, the mathematical model which describes this translation was evolved.

In general, this chapter presented an opportunity for an analysis of some aspects of experimental functions.

## Chapter 4

## AN EXPERIMENTAL APPROACH TO NONLINEAR FUNCTIONS

### 4.1 Introduction

Not all physical situations can be described by a simple straight line. In certain cases the graph of one variable plotted against another will be some sort of curve. Usually these cases can be approached in a fashion similar to that used with linear functions, but new mathematical models must be found.

In this chapter you will learn that nonlinear models are needed to represent certain physical situations. The nonlinear relations that we will encounter here represent a more complicated kind of function than the linear function.

These functions will give you a deeper insight into concepts which have already been introduced.

### 4.2 The Wick: A Classroom Experiment

You have seen and used many examples of materials which absorb water, milk or other liquids. When your little brother spills his milk someone is apt to use a napkin, paper towel or dish cloth to absorb the milk.

Years ago your grandparents probably used kerosene lamps for lighting in their home. Your parents may have a kerosene lamp or lantern for use in camping or at home if the electrical power is cut off. The strip of material which hangs down into the kerosene and extends up to the burner is called a wick. This wick absorbs the kerosene and conducts it to the burner.

No doubt you have observed this phenomenon in a number of situations where a liquid travels along a strip of material.

At what rate does this absorbing take place? Is the rate of travel constant, increasing, or decreasing? Can we build a physical model of this process? How about a mathematical model? An interesting experiment can help you to answer some of these questions.

For this experiment you will need a strip of filter paper or chromatography paper, a container for water, and a watch or clock with a sweep second hand. The purpose of the experiment is to gather data regarding the rate of ascent of water up the paper wick. From this data we will attempt to determine if this movement obeys some physical law. To achieve this we must somehow make some progress readings as the process is going on. Let us take a strip of the chromatography paper 15 cm long and mark it with dots at one centimeter intervals along its length as shown in Figure 1.

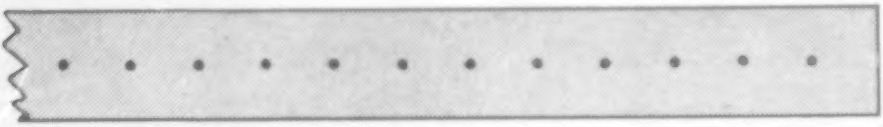


Figure 1

This strip will be your wick. Now slip a regular paper clip on one end and a three-inch piece of transparent tape on the other end as an extension for hanging the strip from a support. Start with the second mark from the paper clip end and number each dot from 0 to 7 inclusive. Now your strip should look like Figure 2.

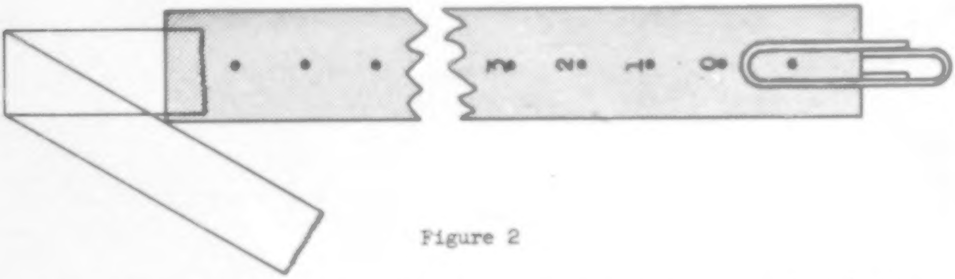


Figure 2

A glass or pyrex 500 cc beaker is ideal for your water container but you could use a regular drinking glass. If it is breezy in your classroom a taller beaker or a quart jar might make your work easier. You need to rig some manner of hanger for the wick. If you use a low container for the water, a stack of books with the top book protruding out about two inches over the ends of the other books will work fine as a place to suspend the wick. Just lower the wick into the water and stick the tape to the edge of the protruding book. If you use a tall jar simply lay a pencil or ruler across the top of the jar and hang the wick from here so that it reaches into the water up to the zero point. Figure 3 shows how these two alternate set-ups might look. Either may prove more satisfactory for you.

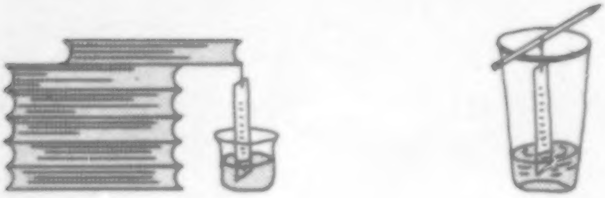


Figure 3

Here are some helpful hints for performing your experiment.

(1) Sight across the water surface and hold the wick on the outside at the proper level so that the zero point of the wick is even with the surface of the water. Do this to determine ahead of time about what level to stick the tape to your hanger.

(2) Work with a partner so that when you dip the wick into the water to the zero point your partner can be ready to watch the water move up the wick and record the seconds passed as the water reaches each successive numbered point. Be ready! It will move fast at the beginning.

(3) Record your data in a table listing the numbers 0 through 7 in one column and the time which corresponds with each point in the other column. Your table might look like the one shown in Figure 4.

| Centimeter mark | Time min - sec | Total seconds expired |
|---|---|---|
| 0 | 8 00 | 0 |
| 1 | 8 08 | 8 |
| 2 | 8 45 | 45 |
| 3 | 9 14 | 74 |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |

Figure 4

From your table of recorded data you can form a set of ordered pairs. Each ordered pair should be of the form (centimeter mark, total seconds). You may need to repeat the experiment several times to improve your technique. When you feel you have gained a useable set of data, you are ready to construct a physical model of your data in the form of a graph. Let your origin be at the lower left hand corner of a full sheet of graph paper. Determine your horizontal and vertical scales based on your data so that you will use the whole sheet of paper for your graph. Plot your set of ordered pairs on the graph paper. You are now ready to investigate the mathematics of the Wick.

### 4.3 The Physical Model

The Wick Experiment has given us a set of ordered pairs of the form (distance, time). The graph should look similar to the graph in Figure 5.

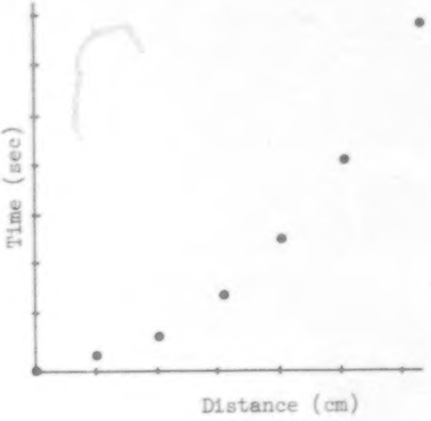


Figure 5

Although these data points are, in themselves, a relation they are not particularly useful in describing the behavior of the Wick. We would like to construct a physical model which would allow us to predict ordered pairs between the points. For any intermediate length in the domain we would like to be able to determine the corresponding time interval. Quite naturally we are inclined to connect the various data points. Our first tendency might be to connect the points by a best straight line. However, it becomes immediately obvious that no straight line can fit the data. In fact, the array of points on the graph carries a strong tendency toward a continually increasing slope. If we connect successive points with straight line segments, we get a model which shows this tendency even better. This model seems to say that the absorption progresses regularly for a short time; then there is a sharp jump, after which it again progresses regularly. In the experiment, we did not observe any jumps in absorption which would account for kinks in the graph. We realize that there should be no particular reason for the kinks to appear at the points which were graphed. If this were the case and we had taken data at half-centimeter intervals there would be an extra set of kinks between the present set of data points. Connecting the data points by segments is possible, but, as we see, not very realistic. A more realistic physical model would be a smooth curve through or near the points. A smooth curve representing the data is shown in Figure 6. Drawing the "best" curve means the smooth curve which you feel fits the data. Even though everyone in the group uses the same data this does not mean everyone will draw the same curve to construct this physical model.

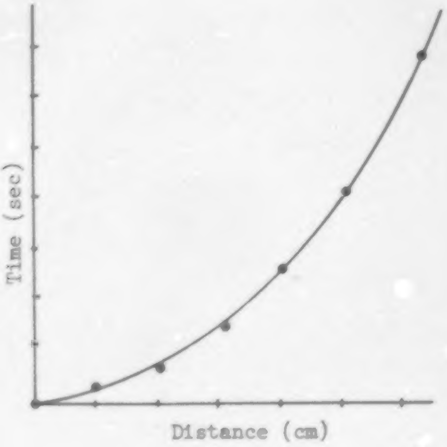


Figure 6

The next step is to see if our physical model leads to a simple mathematical representation. This model should represent the physical situation both accurately and concisely. In addition, we may be able to use the knowledge gained from the mathematical model to help us understand the physical world. The question now is how to proceed. Since we already know something about linear functions, it may be wise to attempt to use this knowledge in the present case.

## 4.4 Mathematical Model

We have generalized the experimental results and through physical reasoning created a physical model.

Since the graph is not a straight line it is immediately evident that it is not linearly related to d . However it may be possible to form ordered pairs by performing some operation on d or t which will result in a linear relation. For example, perhaps one of the following types of ordered pairs will give a linear relation: $(\frac{1}{d},t)$, $(\frac{1}{d^2},t)$, $(d^2,t)$, or $(\sqrt{d},t)$. When a mathematician or scientist has studied many relations and their graphs he is usually able to determine from the shape and location of a graph an approach to the related mathematical model. Remember your work in Chapter 2 and 3. When the data graphed produced a straight line you came to know that an equation of the form $y = mx + b$ could be used to describe the graph.

### Exercise 1

Each of the following problems consists of a set of ordered pairs of the form $(x,y)$.

(a) Graph each set of ordered pairs. (Check the domain and range before setting scales on the x and y axes.)

(b) Draw a smooth curve through the points.

(c) Form a new set of ordered pairs following the instructions given with each problem. (Problem 1 is partially completed as an example.)

(d) Graph this new set of ordered pairs on a new sheet of graph paper.

(e) In each case part (d) should yield a straight line; find the equation of this line using the methods of Chapter 3.

1. $\{(0,0), (\frac{1}{2},\frac{1}{4}), (1,1), (2,4), (3,9), (4,16), (5,25)\}$
   Form ordered pairs of the form $(x^2,y)$.

   $\{(0,0), (\frac{1}{4},\frac{1}{4}), (1,1), \ldots, (25,25)\}$

2. $\{(1,0), (2,6), (3,16), (4,30), (5,48), (6,70)\}$
   Form ordered pairs of the form $(x^2,y)$.

3. $\{(0,1), (1,1\frac{1}{2}), (2,5), (3,14\frac{1}{2}), (4,33)\}$
   Form ordered pairs of the form $(x^3,y)$.

4. $\{(0,3), (1,4), (4,5), (9,6), (16,7), (25,8)\}$
   Form ordered pairs of the form $(\sqrt{x},y)$.

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5. $\{(30,1), (15,2), (10,3), (6,5), (3,10), (2,15), (1,30)\}$
Form ordered pairs of the form $(\frac{1}{x},y)$.

6. Using the set of ordered pairs $(d,t)$ you obtained from the Wick Experiment, form and graph the ordered pairs:

(a) $(d^2,t)$;
(b) $(d,t^2)$;
(c) $(d^3,t)$.

Which of these gives data which is closest to a straight line?

## 4.5 The Horizontal Metronome

Oscillating systems provide a convenient and easily constructed means for generating nonlinear functions. Such functions also occur very often in our everyday life. A point on any rotating wheel exhibits an oscillating behavior. Since an oscillating system repeats itself in time and space, measurements can be started and stopped at convenient times and places. For example, a pendulum can be started and allowed to swing until any irregularities have disappeared. After these irregularities have disappeared the timing can be started and the time for one swing measured. We do not have to initiate the motion and start the timing at the same time. On the other hand for a ball rolling down an inclined plane the timing must be started at the same time the ball is released.

In this experiment we will examine an oscillating system comprised of a hack saw blade clamped in a vise at one end and loaded at the other with a piece of lead. The equipment is illustrated in Figure 7. Clamp the blade so that the motion is in a horizontal plane. When any stiff rod clamped at one end is pulled aside, a force is felt which tends to restore it to its original position. When released the rod will pass through the equilibrium position and the direction of the restoring force will be reversed. Therefore, the rod will exhibit to and fro motion, and we say it is oscillating.

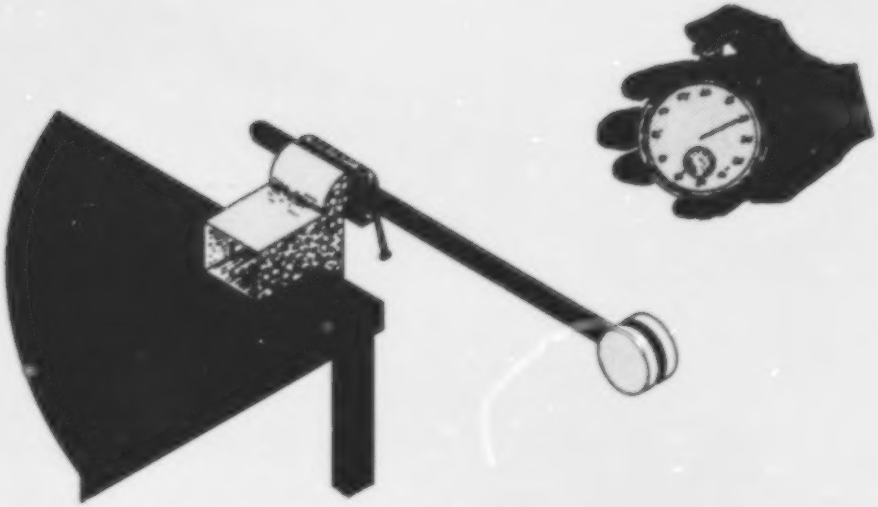


Figure 7

For any type of motion which repeats itself in equal intervals of time, the time interval between any event and the moment the same event occurs again is called the period. The period is usually measured from a point of maximum deflection. For example, the period of the hack saw blade will be the time interval for the lead weight to move from one extreme position until it returns to that same position.

The hack saw blade without the attached mass will vibrate very rapidly. The corresponding period is small. Placing a mass on the free end of the blade will slow the vibrating motion of the blade and thus increase the period.

A little experimentation will also show that the period of vibration depends upon the length of the rod. This length is measured from the edge of the jaws of the vise to the center of the lead mass. If we allow a short length of the rod to vibrate, the period will be small. However the longer the length (d) the longer the period (t). Therefore, the period of vibration depends on the length of the rod. Other physical characteristics can influence the period of the rod. One of these is the size and shape of the rod and another is the maximum displacement of the swing from the rest position (amplitude).

In this experiment we are going to investigate how the length of the rod influences the period. Once this has been decided we must fix all of the other possible variables. Hence, if we take a particular hack saw blade, a fixed mass for the load, and keep the amplitude fairly constant, there should be no influences on the period other than the length.

These conditions and the equipment form a basis for the experiment. For each selected length of the blade (d) we will measure the period (t). To each length there corresponds only one period. Thus the two measurements form an ordered pair (d,t). That is, a length-time relationship. For relations which may be nonlinear, it is advisable to have the domain cover as large an interval as possible. If the blade measures 30 cm, clamp it in the vise to give a starting length of 20 cm. Make a period measurement at this setting. Shorten the blade length by 1 cm and take a new period measurement. Repeat this process of adjusting the blade length and measuring the period until you get a blade length of 10 cm. Below this 10-cm length the period will probably be so short that time measurements by visual methods are impossible. A convenient and more accurate method for determining the period is to take the time for 50 oscillations with a stop watch and the divide this time by 50. This method of measuring period gives a more accurate result than trying to measure the period for a single oscillation. Starting with the longest length and working toward shorter lengths has a definite advantage. Long lengths correspond to long periods and are easy to measure. The techniques developed to measure longer periods will prepare you to measure the smaller periods. You will probably find that periods shorter than 0.5 sec are quite difficult to measure accurately.

The length in centimeters of the blade (d) is measured from the vise jaws to the center of the lead weight. The distance should be measured to the nearest millimeter. Record d in the first column of your data sheet. Use the next two columns to record the number of oscillations and the total time in seconds. From this data calculate the period (t), and record in column four. Your table might look like the one in Figure 8.

| Length d (mm) | Number of Oscillations | Total Time | Period t (sec) | | |
|---|---|---|---|---|---|
| | | | | | |

Figure 8

Now form the ordered pairs (d,t) and plot them on a coordinate plane. If you like, you can record these ordered pairs in another column of your table. Label the horizontal-axis d, and the vertical-axis t. Select the scales for both distance and time so that the graph will come as close as possible to filling the paper. The graph of the distance-period relation will look similar to Figure 9a.

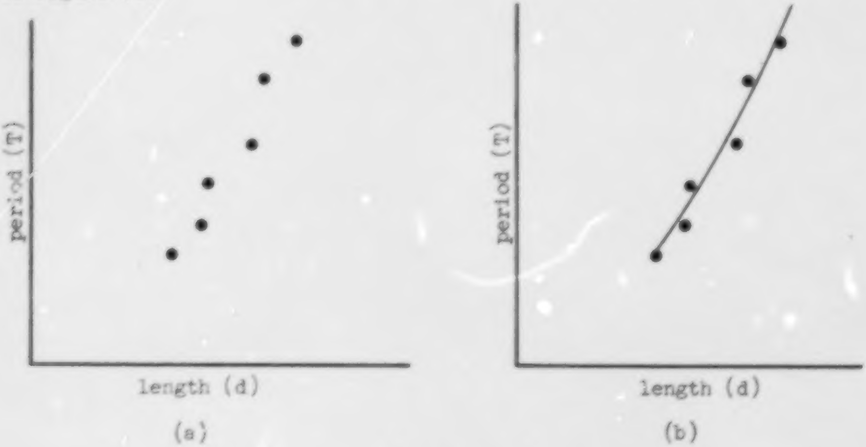


Figure 9

Caution! At first glance this set of points might appear to suggest a linear relation. But a straight line through these points, if extended, would intercept the d-axis to the right of the origin. This would suggest the existence of an arm of some length that could not vibrate.

Actually, we can see from the experiment that as the length of the metronome arm is shortened, the time of the period also gets shorter. This indicates that the graph should approach the origin instead of intersecting the d-axis at some other point.

In Figure 9b we have drawn a segment of a curve through the plotted points. We do not have enough information to extend this curve closer to the origin at the present time.

Here again, as in the Wick Experiment, it is evident that we do not have a linear relation. The graph of our data is not a straight line. You saw in Exercise 1 how it was possible to form a new set of ordered pairs from the data. The graph of the new set of ordered pairs will have a different shape than the graph of the original set.

Consider the possibility that a linear relation may exist between $d^2$ and t. Return to your data page, label a new column $d^2$, and compute the value of

$d^2$ for each measured d. On a fresh sheet of graph paper construct a new set of axes with the scales suitable for the distance squared ($d^2$) and the period (t). Now plot the new ($d^2$,t) ordered pairs. A graph similar to the one you will probably get is shown in Figure 10.

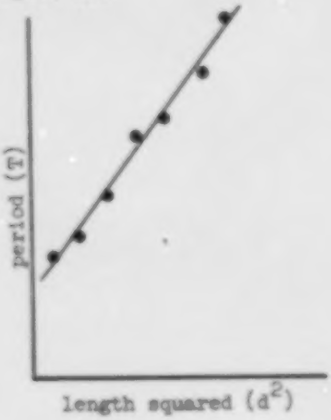


Figure 10

If our guess has been correct these points will fall on a straight line, and we can say that t is linearly related to $d^2$. Calculate the slope of this line and use the point-slope form to determine the equation of the line. This equation will be of the form $t = md^2 + b$. If the ($d^2$,t) plot had not been a straight line, our next step would be to compute $d^3$ and make a graph using ($d^3$,t) pairs. We would examine this graph to see if it gave a linear relation and then proceed to find the linear equation.

We would now like to see if this equation can be used as a mathematical model for the curve in the first graph as drawn in Figure 9b. To check this, select several values from the domain d. Use the equation $t = md^2 + b$ with your values of m and b to calculate the period predicted by the equation. Form a new set of "theoretical" ordered pairs (d,t) and plot these (d,t) points on the same sheet of coordinate paper used for your experimental points. Common practice is to use solid circles to indicate the "theoretical" points and open circles for "data" points. Use a dashed line to draw a smooth curve through the solid circles so you do not confuse this new curve with the original curve through the open circles. If this curve compares favorably to the experimental curve then we can use the equation as our mathematical model of the metronome.

The domain, as defined by the experiment, did not encompass all points of physical interest. Since we could not measure short periods, it would be of value to try to predict them. Great care must be exercised in doing this. It is obvious from the physical construction that before zero length is reached the weights will prevent further shortening of the blade. In the other direction it is obvious that continued extension of the blade is impossible. With these limitations in mind the extension can be carried out in the following way. The mathematical model of the relation between d and t predicts a period for each blade length including zero. Values of d within the extended domain are selected and the corresponding periods (t) calculated. These new values of d and t are now plotted on the original graph and the curve extended. Of course t can also be read directly off the straight line graph of $(d^2,t)$.

## Exercise 2

1. The following equations describe various curves.

   (a) What form of ordered pairs would you predict in each case to show a straight line graph?

   (b) Use the following numbers (-2,-1,1,2) from the domain of the given relation to form the predicted ordered pairs.

   (c) Plot these points and check to see if they fall in a straight line.

   (d) Write the linear equation for each graph.

Example: $y = 3\left(\frac{1}{x}\right) + 2$

Predict ordered pairs of the form $\left(\frac{1}{x},y\right)$.

$y = 3\left(-\frac{1}{2}\right) + 2$

$= -\frac{3}{2} + 2 = \frac{1}{2}$ $\qquad\qquad \left(-\frac{1}{2}, \frac{1}{2}\right)$

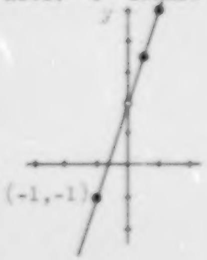
In a similar way the following ordered pairs are calculated:

$(-1,-1)$

$(1,5)$

$\left(\frac{1}{2}, \frac{7}{2}\right)$

Note: 0 cannot be used to form an ordered pair for this relation since $\frac{1}{0}$ is undefined.

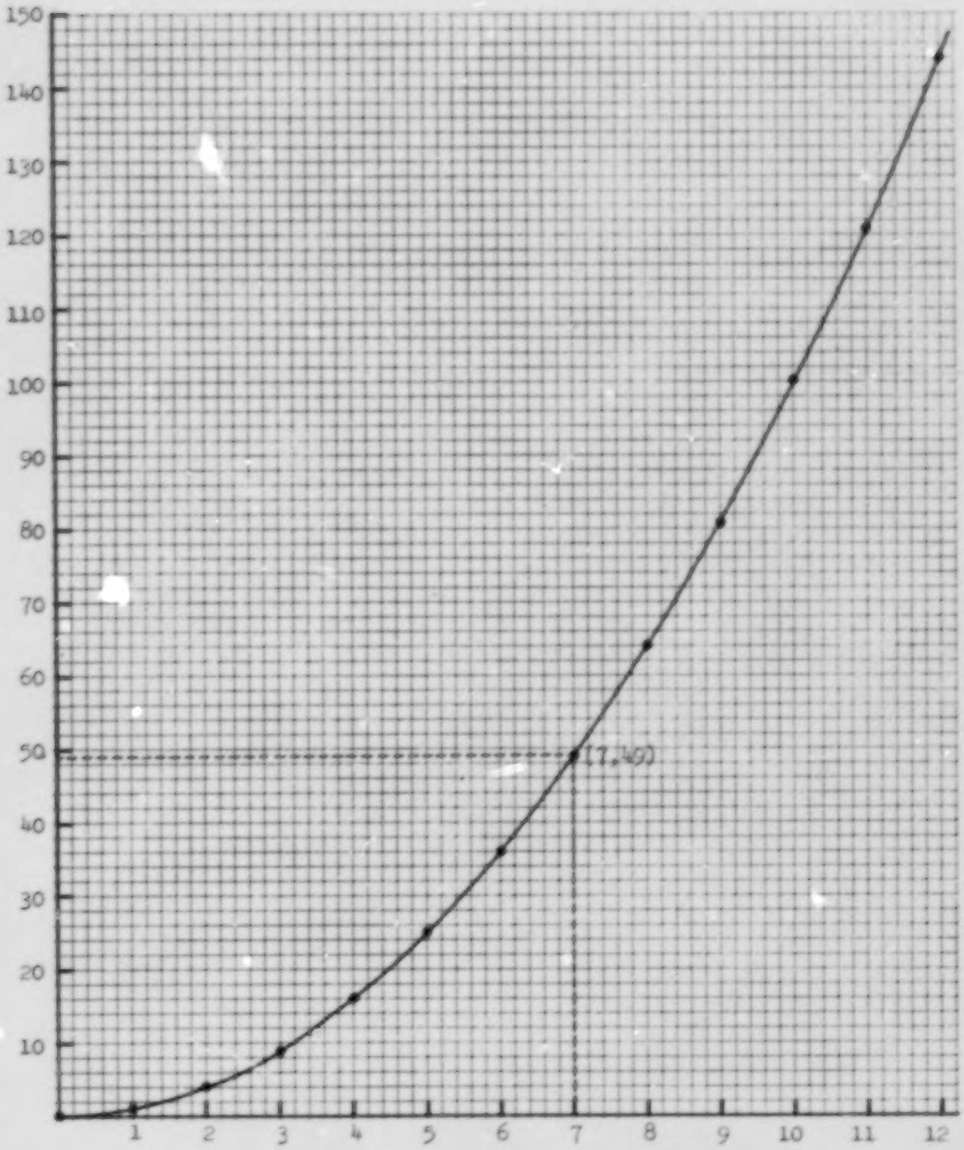


This point is missing from the graph since the ordered pair (0,2) is not in the relation.

The equation of the line is $y = 3u + 2$. The domain of u is all real numbers except 0 and range of y is all real numbers except 2.

(a) $y = x^3 + 7$ (b) $y - 3 = x^2 + 7$ (c) $y = \frac{1}{x^2} - 4$

2. If you pick any point on the graph of $y = x^2$, the first element of the ordered pair will be the square root of the second element.



For example, to find $\sqrt{49}$, consider the ordered pair of the graph for which 49 is the second element (7,49). The first element of this ordered pair is 7 which is the $\sqrt{49}$ .

From the graph obtain the following values.

(a) $\sqrt{25}$
(b) $\sqrt{121}$
(c) $\sqrt{2}$
(d) $\sqrt{40}$
(e) $\sqrt{68}$
(f) $\sqrt{8}$
(g) $\sqrt{116}$
(h) $\sqrt{74}$

(i) $(4.6)^2$
(j) $(9.4)^2$
(k) $(10.8)^2$
(l) $(3.2)^2$
(m) $(5.5)^2$
(n) $(6.8)^2$
(o) $(76)^2$
[Hint: $(76)^2 = (7.6 \times 10)^2$
$= (7.6)^2 \times 100$]

3. From your original graph of (d,t) pairs find the value of t corresponding to d = 8.5 cm. Using the equation you obtained to describe the distance-period relation, calculate the period corresponding to a distance of 8.5 cm. Compare the two results.

4. Each of the following sets of ordered pairs (d,r) describe various curves.
   (a) Plot the points.
   (b) Draw the curve.
   (c) Form new ordered pairs of the form $(d^2,r)$ and plot these points.
   (d) If the $(d^2,r)$ ordered pairs form a linear relation draw the straight line and find the equation of the line.

(a) $\{(0,0), (1,2), (2,8), (3,18), (\frac{1}{2}, \frac{1}{2})\}$

(b) $\{(0,2), (1,3), (2,6), (3,11), (4,18)\}$

(c) $\{(0,2), (2,0), (1, \frac{3}{2}), (\frac{1}{2}, \frac{15}{8})\}$

(d) $\{(1,1), (2,10), (3,25), (\frac{3}{2}, \frac{19}{4}), (\frac{5}{2}, \frac{67}{4})\}$

(e) $\{(1,0), (\frac{3}{2}, \frac{19}{8}), (2,7), (3,26), (\frac{5}{2}, \frac{117}{8})\}$

5. If we consider the domain of d to include all positive real numbers, use your mathematical model to calculate the values of the period that correspond to the following values of d.

(a) d = 50 cm
(b) d = 100 cm

(c) d = 500 cm
(d) d = 1000 cm

## 4.6 The Parabola

In the chapter on linear functions we learned that coefficients such as $m$ and $b$ in an equation of the form $y = mx + b$ could be used to describe a line. We now have an equation of the form $y = mx^2 + b$ , and we are using this equation to describe a curve. The $m$ in this case is not the slope. Even though $m$ is a number, the ratio of "rise" to "run" for our curve is not constant. Since we have reserved these letters, $m$ and $b$ , to refer to properties of a line let us change the notation in this new equation so the constants will not be confused with the slope and y-intercept of a straight line. Your equation with numerical values for $m$ and $b$ will of course not be changed. We will now use the letter $A$ to refer to the coefficient of the $x^2$ term and the letter $C$ as the constant term. The equation will now read

$$y = Ax^2 + C .$$

Equations of the form, $y = Ax^2 + C$, where A and C are real numbers and A is not 0, are called quadratic equations.

To investigate the influence of $A$ upon the curve we can set $C$ equal to zero and then determine the shape of the graph of $y = Ax^2$ for different values of $A$. The tables in Figure 11 give ordered pairs for various equations of the form $y = Ax^2$. The graphs of these ordered pairs are shown in Figure 12.

Table I

| $y = x^2$ | |
|---|---|
| x | y |
| -4 | 16 |
| -3 | 9 |
| -2 | 4 |
| -1 | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| 4 | 16 |

Table II

| $y = \frac{1}{2}x^2$ | |
|---|---|
| x | y |
| -4 | 8 |
| -3 | $\frac{9}{2}$ |
| -2 | 2 |
| -1 | $\frac{1}{2}$ |
| 0 | 0 |
| 1 | $\frac{1}{2}$ |
| 2 | 2 |
| 3 | $\frac{9}{2}$ |
| 4 | 8 |

Table III

| $y = -\frac{1}{2}x^2$ | |
|---|---|
| x | y |
| -4 | - 8 |
| -3 | $-\frac{9}{2}$ |
| -2 | - 2 |
| -1 | $-\frac{1}{2}$ |
| 0 | 0 |
| 1 | $-\frac{1}{2}$ |
| 2 | - 2 |
| 3 | $-\frac{9}{2}$ |
| 4 | - 8 |

Table IV

| $y = 2x^2$ | |
|---|---|
| x | y |
| -3 | 18 |
| -2 | 8 |
| -1 | 2 |
| 0 | 0 |
| 1 | 2 |
| 2 | 8 |
| 3 | 18 |

Table V

| $y = -2x^2$ | |
|---|---|
| x | y |
| -3 | -18 |
| -2 | -8 |
| -1 | -2 |
| 0 | 0 |
| 1 | -2 |
| 2 | -8 |
| 3 | -18 |

Figure 11

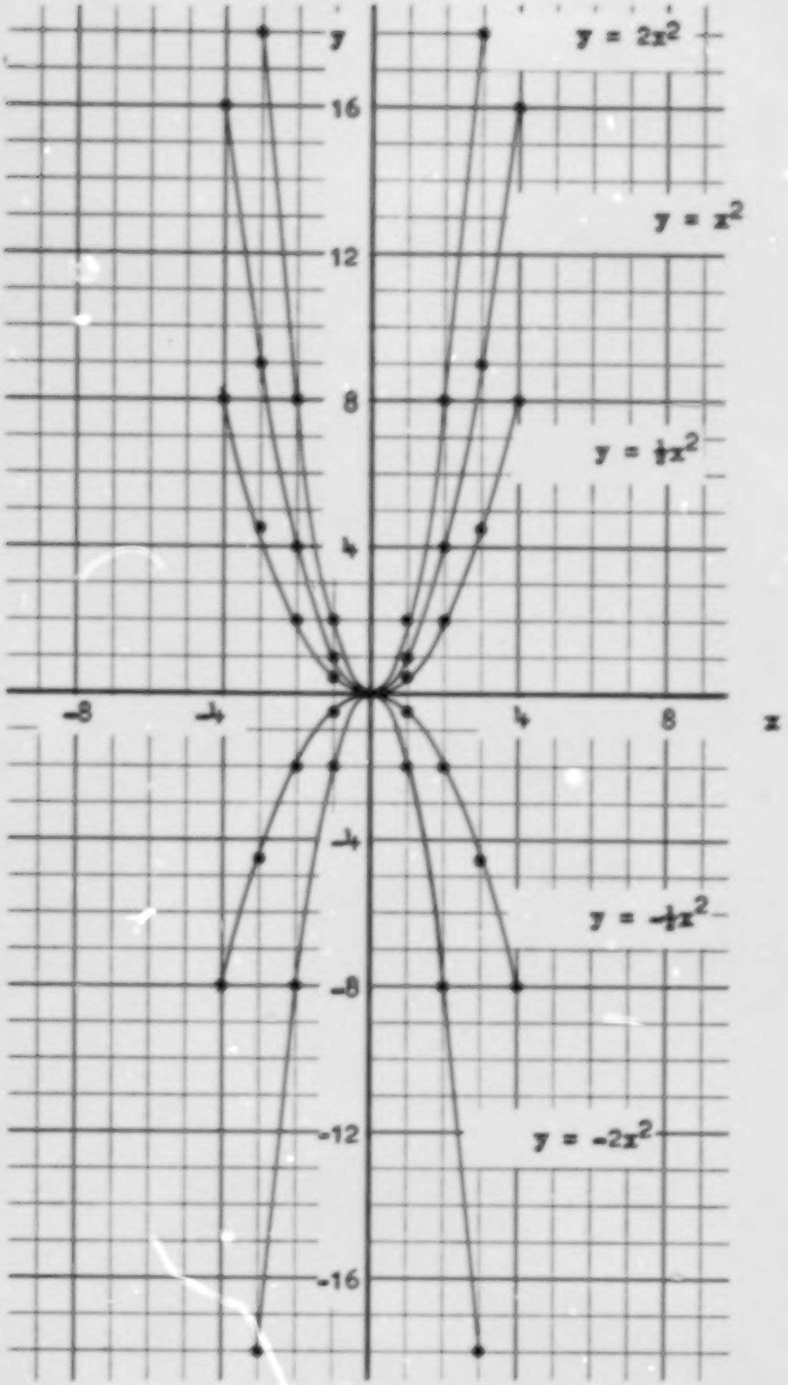
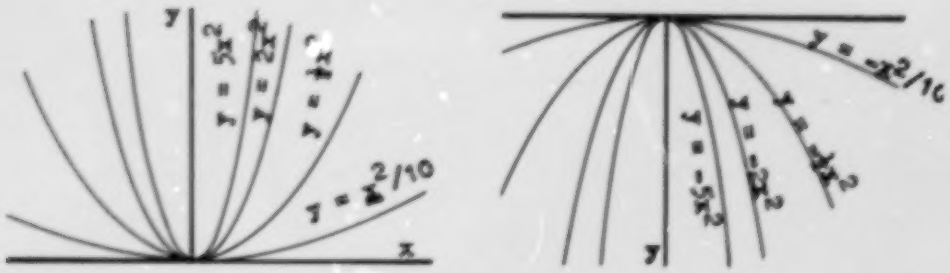


Figure 12

Once we leave an experimental situation and have a purely mathematical relationship it is common practice to let the domain consist of all real numbers which will yield real numbers for the range. For all real numbers A and x, $Ax^2$ will be a real number so the domain of $y = Ax^2$ is the set of all real numbers. This is the reason for the negative values of x used in the table. If the relation had been of the form $y = \frac{A}{x^2}$ the domain would be the set of all real numbers excluding zero. In these tables, x assumes both positive and negative values and the coefficient A has five different values.

From the tables and the graphs of the ordered pairs we can see that there is a definite analogy between the m in the $y = mx$ and the A in $y = Ax^2$. When the slope m is positive we have a "rising" line and when the slope m is negative a "falling" line. When A is positive in $y = Ax^2$ the curve opens "up" and when A is negative it opens "down". For a line $|m|$ tells us how fast the line rises or falls. The numerical value of A tells us about the "flatness" of the curve. Smaller values of $|A|$ correspond to the "flatter" curves (Figure 13, (a) and (b)).



(a) (b)

Figure 13

Curves of the type we have shown are examples of a type of curve known as a parabola.

We must now consider the influence of the constant term C on the graph of the parabola whose equation is $y = Ax^2 + C$. Notice that for $C = 0$ the curve will pass through the origin. Figure 14 shows the graphs of five quadratic relations of the form $y = Ax^2 + C$. The value of A is one for each relation but the C has been allowed to vary.

The graphs shown in Figure 14 are only sketches of the relations. If you make tables of ordered pairs for each of them and plot them carefully on a sheet of coordinate paper you can probably see that all of the curves are the same size and the same shape. Take a sheet of onion-skin paper and place it on the coordinate paper. Copy the graph of $y = x^2$ directly on the onion-skin paper and then move this curve until it coincides with each of the other curves. This method will show you that the graph of $y = x^2 + 2$ is exactly like that of $y = x^2$ except for its placement on the coordinate plane.

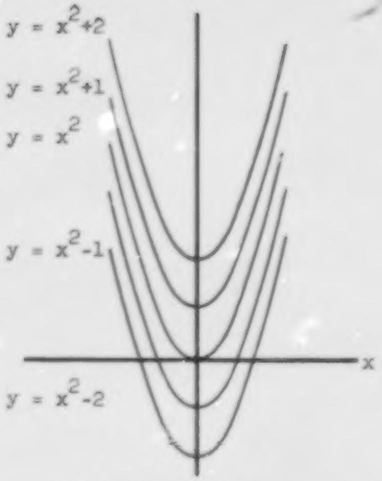


Figure 14

The graphs of $y = -x^2 + C$ where $A = -1$ and $C$ takes on various values are shown in Figure 15.

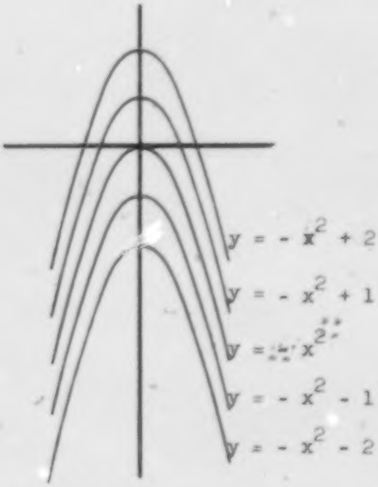


Figure 15

When we set $A = 1$, all of the parabolas opened upward and each parabola had a "lowest" point which we will call the minimum point. When A had the value of -1, the parabolas were inverted so that they opened downward. Each of these parabolas had a "highest" point or *maximum point*. We have already concluded that the $|A|$ tells us about the "flatness" of the parabola. Now we conclude that if $A > 0$ the parabola opens upward and has a minimum point, and if $A < 0$ the parabola opens downward and has a maximum point. For all parabolas which are graphed from quadratic equations of the form $y = Ax^2 + C$ this maximum or minimum point is called the *vertex of the parabola*.

Now let us look back at our graphs and see what we can conclude about the effect of the constant term C upon the graph of the parabola. The C in $y = Ax^2 + C$ does not have any effect on the shape of the curve but does tell where the vertex of the graph will lie. For example, in the graph of the equation $y = x^2 + 2$ the minimum point of the parabola was at the point where the graph intersected the y-axis and the coordinates of the vertex were (0,2). The graph of the equation $y = -x^2 - 1$ had a maximum point where the parabola intersected the y-axis and the coordinates of this vertex were (0,-1).

You might ask if the vertex must always lie on the y-axis. The answer to this question is "no". However, all of the parabolas we will study in this chapter will have their vertex on either the y-axis or the x-axis.

## *Exercise 3*

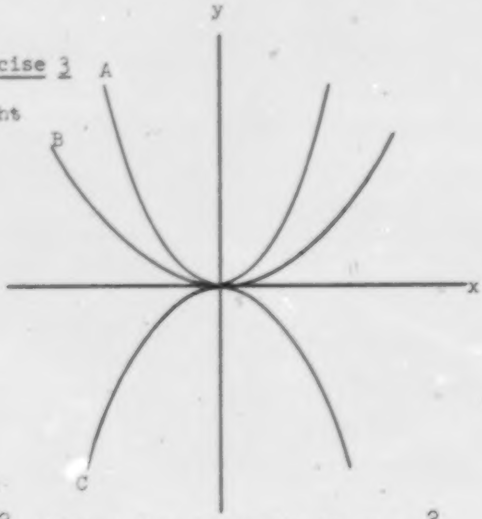
1. The three curves shown at the right are sketches of the graphs of:

   $y = \frac{1}{2}x^2$

   $y = -\frac{1}{3}x^2$

   $y = \frac{1}{6}x^2$ .

   Match each curve with the proper equation.



2. Describe how the graph of $y = Ax^2$ differs from the graph of $y = x^2$ in each of the following cases.

   (a) $A = 0$
   (b) $0 < A < 1$
   (c) $A > 1$
   (d) $A = -1$

3. Make a table of at least seven ordered pairs for each of the following equations. Use both positive and negative values of x. Draw all of the graphs on the same sheet of coordinate paper and label each.

(a) $y = 2x^2$ (c) $y = -3x^2$

(b) $y = \frac{1}{5} x^2$ (d) $y = -\frac{1}{10} x^2$

4. Plot the ordered pairs given below and draw a smooth curve through the points.

| x | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
|---|---|---|---|---|---|---|---|
| y | -3 | -2 | -1 | 0 | 1 | 2 | 3 |

Is this relation a function? Is the converse of this relation a function? Can you think of an equation to describe the relation?

5. For each of the following pairs of equations below, plot the graphs using a single set of coordinate axes for each pair.

(a) $y = 2x^2 + 3$ (d) $y = -x^2 + 1$
$y = 2x^2 - 3$ $y = x^2 + 1$

(b) $y = \frac{1}{2} x^2 + 3$ (e) $y = -2x^2 - 1$
$y = \frac{1}{2} x^2 - 3$ $y = 2x^2 - 1$

(c) $y = -2x^2 + 3$ (f) $y = -3x^2 + 1$
$y = -2x^2 - 3$ $y = 3x^2 + 1$

6. Which of the relations in Problem 5 have a minimum value and which have a maximum value? What are these values?

7. The following equations describe curves which are not parabolas. What ordered pairs would you form in each case to show a parabolic relation?

(a) $y = 2x^4 + 3$ (b) $y = x^6 - 2$

### .7 The Oscillating Spring

This experiment will extend our knowledge of quadratic relations. In xamining the behavior of the Horizontal Metronome we found that the length of he blade and the period of oscillation were connected by a quadratic relation f the type $t = md^2 + b$. This particular form of the quadratic relation was ictated by our experimental apparatus and its design. The domain of the relaion was the set of d values and the range of the relation the set of t values.

In many cases, the relation generated by the experiment is not a quadratic. However, the converse of the relation may be a quadratic. In this situation, the method of obtaining the mathematical model from the data must be altered. The present experiment will illustrate this. We will investigate the role played by the converse of the relation which arises from an analysis of the experimental data.

Springs are simple mechanical devices found most everywhere. Those designed to be squeezed together are called compression springs. Those meant to be stretched are called tension springs. A tension spring may be made to perform in an unusual way as follows. Suspend the spring in a vertical position and hang a mass at the lower end. After pulling the mass downward and releasing it, the mass and spring will oscillate up and down over and over again, for a time of several minutes. The general arrangement is shown in Figure 16.

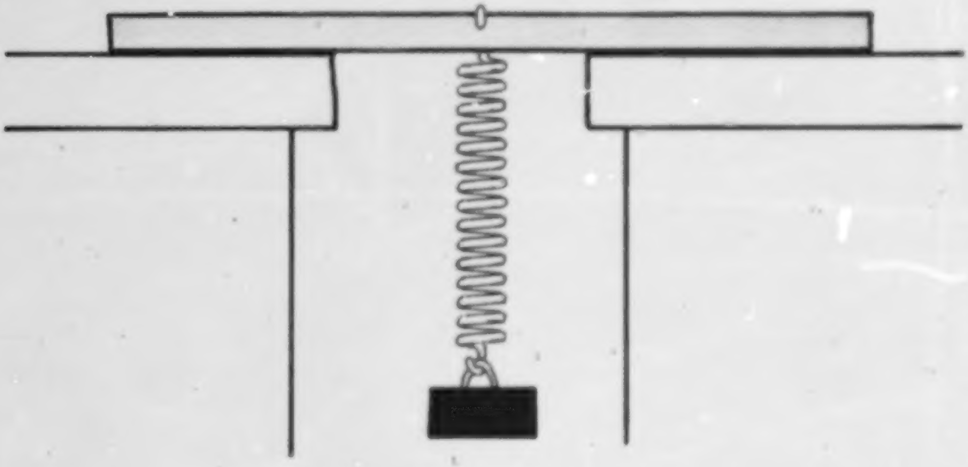


Figure 16

One important variable in this situation suggests itself immediately. It is the period of the oscillation. You will recall that if the motion repeats itself in equal time intervals, this time interval between any event and the moment that same event occurs again is called the period. For the oscillating spring the period can be determined by recording the time between successive trips of the mass to its lowest point.

As always, all the possible variables which could conceivably influence the period must be listed and examined. Some of the possible influences upon the period are: the size and type of spring used; the mass of the object suspended from the spring; the total distance through which the mass swings from one extreme to the other; the temperature of the spring. One, and only one, of these influences must be allowed to change during the experiment. The other variables must be kept constant so that whatever their influence upon the period may be, it will not be changed during the course of the experiment. Therefore the period will depend on the physical quantity we let vary.

Only a little experimentation is required before discovering that a change in the mass suspended from the spring has a decided influence upon the period of oscillation of the spring. This does not mean that the other variables which are held constant do not influence the period, but only that these will have the same influence upon the period during the experiment.

When a variety of masses are used on the spring there corresponds a definite value of the period to each mass. Two columns of data are needed, one for the mass and the other for the period. It will be convenient to use 100, 200, 300, ..., 1000 gram masses, thus providing ten load values ( $\ell$ ) of the domain. If standard masses are not used, the masses of the objects that are used should be measured in advance. A single period is not easy to measure. Fifty consecutive periods, however, are easily timed with a stop watch. When this time interval is divided by 50, the time of a single period is obtained.

The appropriate columns of data are as follows: The first column for the mass in grams ( $\ell$ ), the second column for the number of oscillations, the third column will show the total time in seconds, and the fourth will list the period (t). Each column should have its appropriate heading. Pattern your table after the table illustrated in Figure 8.

The collection of mass-period pairs ($\ell$,t) shown in the table is a relation. As with linear relations, much may be learned by graphing the relation on coordinate paper. Since it has been decided in this experiment to let the set of masses be the domain of the relation, mass values ( $\ell$ ) should be plotted along the horizontal axis. The range of the relation, the period (t), should be plotted along the vertical axis.

## 4.8 The Physical Model

Once the data pairs are plotted on coordinate paper, let us seek the simplest possible model that will describe the behavior of the oscillating spring. As before, we are inclined to connect the points in some fashion. If straight line segments are drawn from point to point, we are assuming something about the behavior of the spring for masses intermediate to the values actually employed in the experiment. That is, we are assuming the relation is linear between points. If, on the other hand, a smoothly changing curve is drawn through or near the points, we are asserting a different behavior for the spring for intermediate mass values. Our physical intuition may tell us that in all probability the smoothly changing curve is the best model. Whether this leads to a simple mathematical model or not remains to be seen.

As before, the drawing of a single curve through or near the points takes account of certain experimental inaccuracies in the data. Experimental inaccuracies may cause a slight displacement of a point one way or another. The desired curve should go through or near the points as smoothly as possible. The smoothness requirement arises only from our feelings about the physical situation. Your graph of the period plotted against load should look similar to Figure 17.

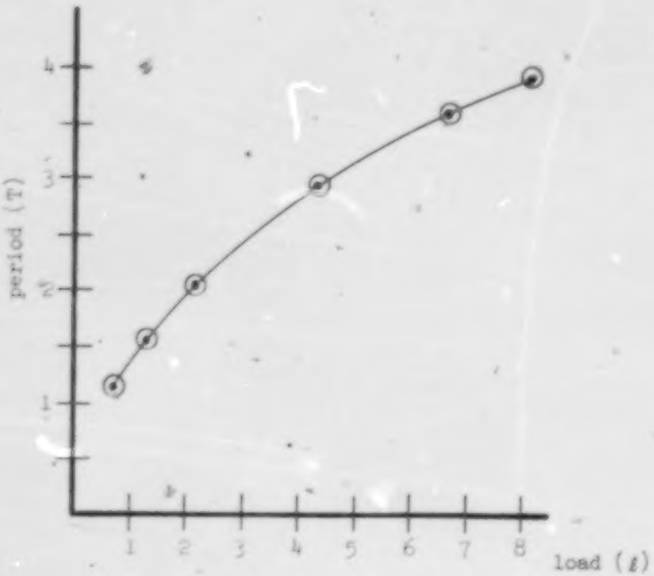


Figure 17

At this point as in the Horizontal Metronome we will attempt to develop a mathematical model (equation) which represents the physical case accurately. Our results clearly indicate that the period (t) is not a linear function of the load ( $\ell$ ). Can we find a new variable related only to the load ( $\ell$ ) which is a linear function of the period (t) ? Since we are looking for a simple combination of $\ell$ 's which when plotted against the period will give a straight line, let us try the solution which was successful in the metronome case. Hence we will try a plot of the ordered pairs consisting of the load squared and the period, $(\ell^2, t)$. The collection of values of $\ell^2$ is the domain and is plotted along the horizontal axis. The period is plotted along the vertical axis. If a line can be found to represent this new graph in a reasonable way, we can state with assurance that t will be linearly related to $\ell^2$. Your plot should look similar to the one in Figure 18.

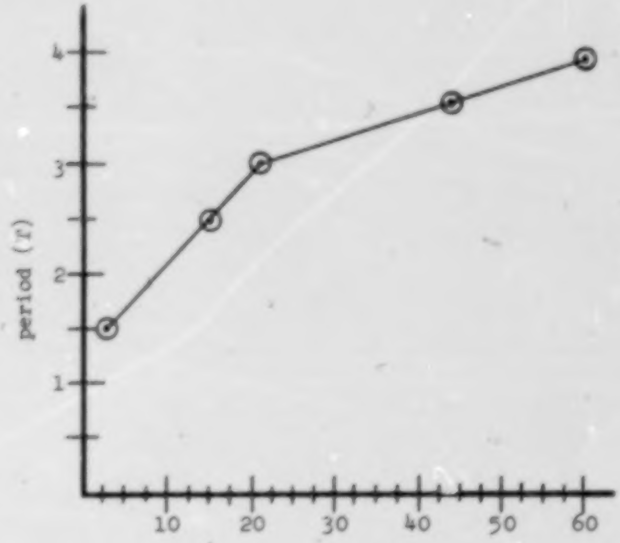
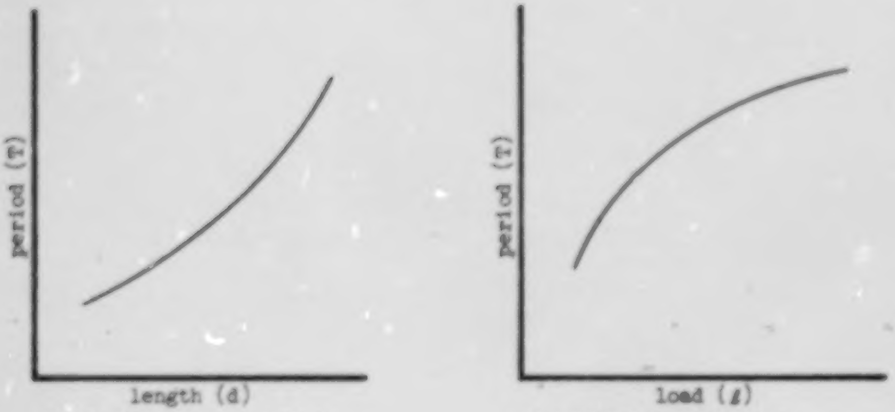


Figure 18

It is immediately evident from our graph that the load squared is not linearly related to the period. That is $t \neq m\ \ell^2$ . Our first guess has led us down a blind alley. The situation is more complex than we at first suspected.

## 4.9 The Oscillating Spring Converse Relation

The problem now is how do we proceed from here. It is obvious that using higher powers n of $\ell$ will only give us a greater bending when we plot ($\ell^n$,t) ordered pairs. Let us see if we have missed something by looking at our graphs of (d,t) and ($\ell$,t) from the two experiments. Similar graphs are plotted side by side in Figure 19.



Horizontal Metronome
a

Oscillating Spring
b

Figure 19

The immediate difference is that the Horizontal Metronome graph (Figure 19a) is bent so that it opens upward. In our present experiment the curve bends so that it opens down. In other words, the two graphs are of the same approximate shape but they are oriented differently with respect to the coordinate axes. Is it possible that this different orientation could be the factor we have overlooked?

The only experience we have had with re-orienting curves was in the chapter on Falling Spheres when we discussed relations and their converses. There our relation was a straight line and its converse was also a straight line oriented differently towards the horizontal and vertical axes. A graph of a straight line and its converse are illustrated in Figure 20.

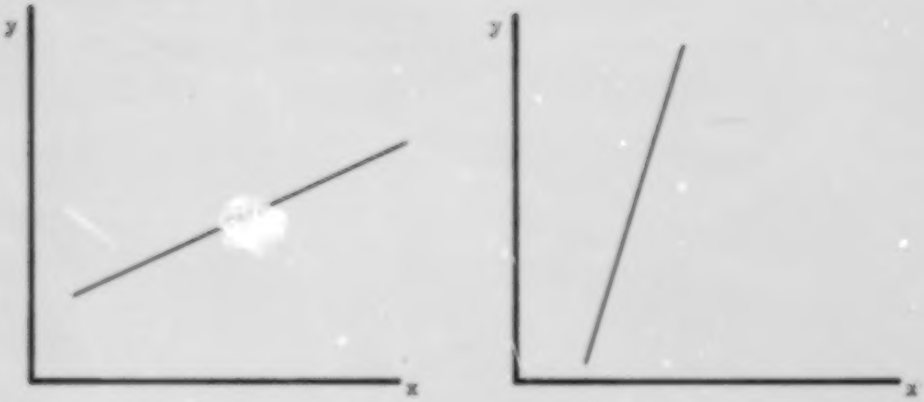


Figure 20

If we construct the converse of our ($\ell$,t) graph it is possible that the new orientation will be similar to the graph found for describing the Horizontal Metronome. We know that in the case of the Horizontal Metronome a linear relation was found between the square of domain elements and the corresponding elements of the range. Hence we may be able to find a simple relation which will describe the converse in the present experiment. There is, of course, no guarantee this will work, but it is worth a try.

A simple and direct method for finding the converse relation is to exchange the elements of the domain for the elements of the range. To generate the converse in the present experiment plot the period (t) data along the horizontal axis and the load ($\ell$) data along the vertical axis. The new graph will consist of period-load pairs (t, $\ell$). A plot of this new relation is illustrated in Figure 21. It is the graph of the converse of the original relation. You will note that interchanging the order of the data pairs interchanges the axis labels also, since they refer to the physical situation.

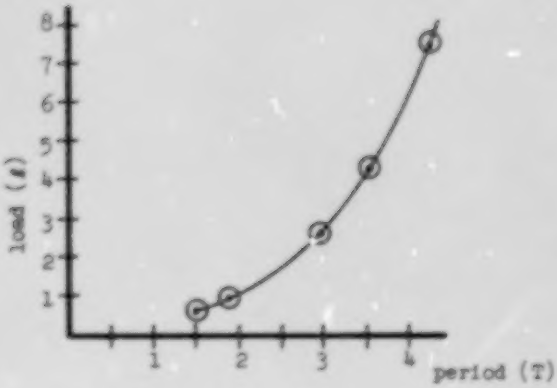


Figure 21

We immediately recognize that this graph has an orientation very close to the graph of Figure 19a which opens in an upward direction. We now proceed in our analysis by plotting a third relation composed of ($t^2$, $\ell$) pairs. A new column of your data sheet should be headed "period squared" or $t^2$, and the appropriate values calculated for each of the original values of t. Now plot the new ($t^2$, $\ell$) points on a fresh sheet of coordinate paper.

If a line can be found to represent this new graph in a reasonable way we can state with assurance that $t^2$ will then be a linear function of $\ell$. Utilizing the slope-intercept expression for a line we may then write in general terms that $\ell = mt^2 + b$. Here as before, m is the slope of the line and b is the intercept with the vertical axis. If you are satisfied that the plotted points can be represented by a line, the expression $\ell = mt^2 + b$ is the equation of the line.

We now have to determine if the equation of the line can also be used to represent the (t, $\ell$) graph of our original data. Make a new column on your data sheet, "load ($\ell$) in grams -- calculated". Use values for the period actually obtained in the experiment and insert in the formula

$$\ell = m \cdot t^2 + b$$

and compute the associated values of $\ell$. Your values of m and b should be used in the equation.

Graph the ordered pairs formed by the (period,calculated load) relation on the same sheet of coordinate paper as your ($t$, $\ell$) graph. Use solid circles to mark these calculated points. Connect the calculated points with a "dashed" curve. If the mathematical model is a good one this new curve should have a close resemblance to the experimental curve. We still have to consider any restrictions on the domain. The mathematical model will give us the load necessary for any period we desire, however the spring may not be capable of supporting such a load. Once we enter the world of generalizations we can extend the domain to include all real numbers, but as long as we remain in an experimental setting, our domain is definitely limited by the equipment being used.

## Exercise 4

The table at the right shows the experimental data for a new oscillating spring. The load ( $\ell$ ) in grams was fixed, and then the corresponding periods ($t$) in seconds were measured.

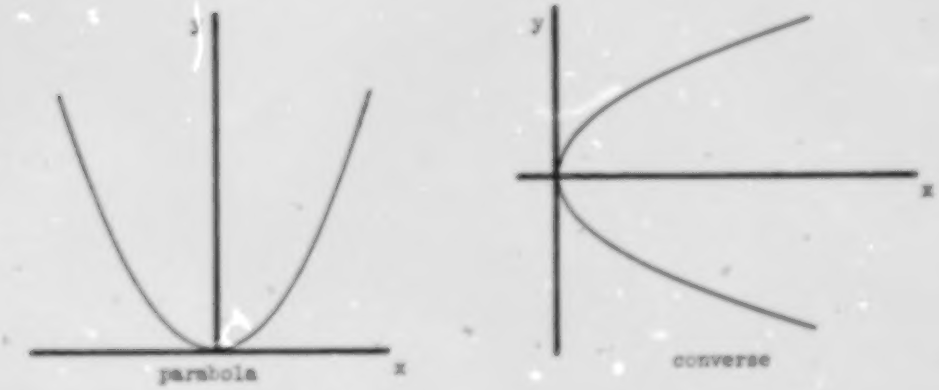
| $\ell$ (grams) | $t$ (sec) |
|---|---|
| 2.5 | 1 |
| 4.0 | 2 |
| 6.5 | 3 |
| 10.0 | 4 |
| 14.5 | 5 |
| 20.0 | 6 |
| 26.5 | 7 |
| 34.0 | 8 |

1. Graph the relation and its converse on separate sheets of coordinate paper.
2. Graph the ($t^2$, $\ell$) relation. Draw the "best" straight line and obtain the equation for $\ell$.
3. Use your equation obtained above to calculate values of the load in grams for each value of the period in the range of the experimental relation. Compare the calculated and experimental values of the load.

## 4.10 Relations and Converses

It was shown that our choice of order for $t$ and $\ell$ had yielded the converse of the parabola. It is very rarely apparent at the beginning of an experiment which order will yield the most direct path to the mathematical model. The order makes no difference in the linear case, but it often complicates our efforts to find expressions for nonlinear relations. The converse can in many cases simplify our search for a mathematical model.

A graphical representation is probably the most helpful means of recognizing relations and their converses. The complete parabola and its converse are pictured in Figure 22.



parabola

converse

Figure 22

The converse is obtained by interchanging the domain and range of the relation. The new domain is still plotted along the horizontal axis and the new range along the vertical axis.

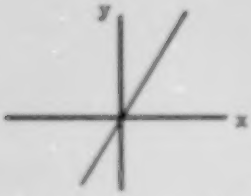
There is one other important point to be made. The mathematical models we have developed in the last two sections are more than relations connecting two variables. In every case each element of the domain has associated with it exactly one element in the range. Each length of the metronome blade and each mass on the spring yielded only one period and each distance on the wick had only one time interval. The single-valued nature of these mathematical models puts them in the class of relations called functions. The full parabola on the left side of Figure 22 is an example of a function. Each value of x has associated with in only one value of y. The converse relation of

Figure 22, however, is not a function. Each value of x has two values of y associated with it.

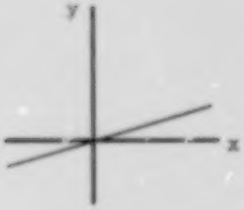
### Exercise 5

1. In the series of graphs shown in the figure on the following page, pair each graph with another so that in each case you have a relation and its converse.
2. Which of the graphs in the figure represent functions?
3. Which pairs of graphs obtained in Problem 1 represent one-to-one functions? (Note: If both a relation and its converse are functions, then these two relations are called one-to-one functions.)

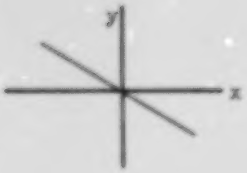
(a)



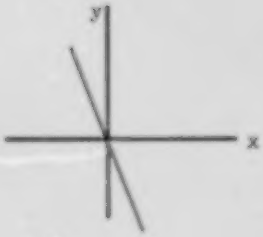
(b)



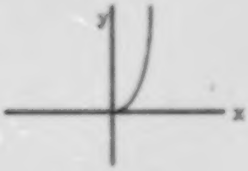
(c)



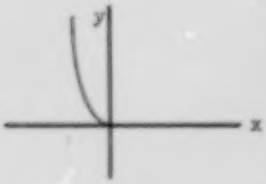
(d)



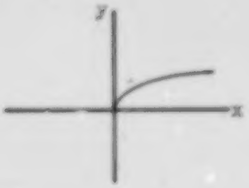
(e)



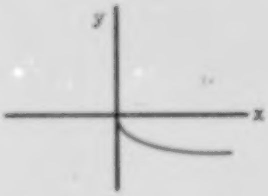
(f)



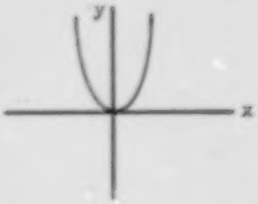
(g)



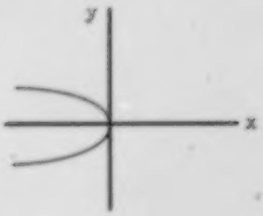
(h)



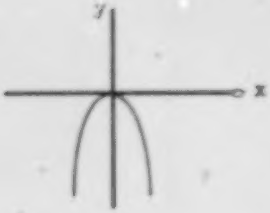
(i)



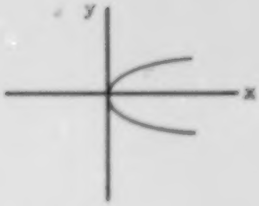
(j)



(k)



(l)



## 4.11 Translation of the Parabola

We have already discussed equations of the form $y = Ax^2 + C$ and have seen how the coefficient of the $x^2$ term determines the shape of the curve and the C term translates the curve up or down. In this section we will study relations defined by equations of the form

$$y = A(x - k)^2$$

where A and k are nonzero constants. As an example, let us draw the graph of

$$y = 2(x - 3)^2 .$$

Let the domain be the set of all real numbers. A table of values and a sketch of the graph are shown below (Figure 23).

| $x$ | ... | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y = 2(x - 3)^2$ | ... | 8 | 2 | 0 | 2 | 8 |

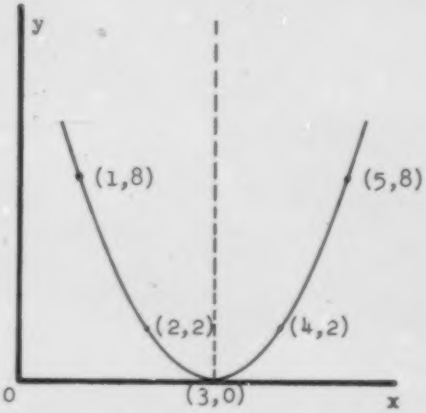


Figure 23

This graph is shaped like the parabolic relation we have been studying, except that the vertex is not on the y-axis. In Figure 24 the graph of $y = 2(x - 3)^2$ is compared with that of $y = 2x^2$.

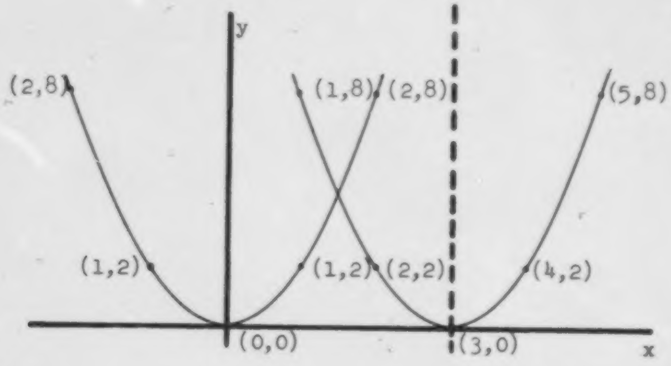


Figure 24

Use a sheet of onion-skin paper and copy the graph of $y = 2x^2$. Shift this paper so that the graph of $y = 2x^2$ is over the graph of $y = 2(x - 3)^2$. The two graphs will be congruent. That is, the graph of $y = 2(x - 3)^2$ is the same as the graph of $y = 2x^2$ but is 3 units to the right. In the same way we could verify that the graph of $y = \frac{1}{2}(x + 2)^2$ is 2 units to the left of the graph of $y = \frac{1}{2}x^2$ and has the same shape as $y = 2x^2$

If we draw the graph of the equation of $y = 2(x - 3)^2 + 2$ and compare it with the graph of $y = 2(x - 3)^2$, we see that the shape of the graph has not changed (Figure 25).

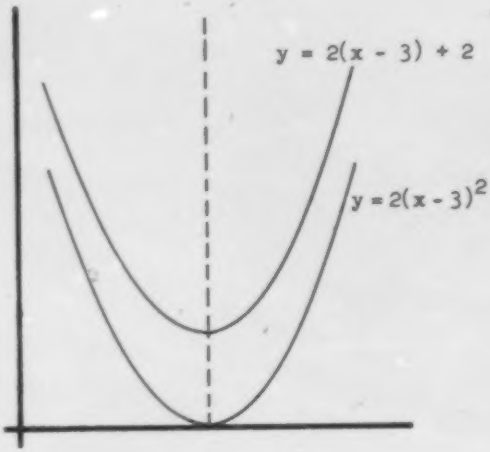


Figure 25

The graph of $y = 2(x - 3)^2 + 2$ is obtained by moving the graph of $y = 2(x - 3)^2$ upward 2 units. Similarly, we can show that the graph of $y = 2(x + 2)^2 - 3$ can be obtained by moving the graph of $y = 2(x + 2)^2$ downward 3 units.

Finally we recall that the graph of $y = 2(x - 3)^2$ is the same as the graph of $y = 2x^2$ shifted to the right 3 units. From this we can see that the graph of $y = 2(x - 3)^2 + 2$ is the same as the graph of $y = 2x^2$ by "moving the graph" to the right 3 units and upward 2 units.

In later courses you will learn that it is always possible to obtain the graph of

$$y = A(x - h)^2 + k$$

from the graph of

$$y = Ax^2$$

by moving the graph of $y = Ax^2$ horizontally $h$ units and vertically $k$ units.

## Exercise 6

1. For each of the following, describe how you can obtain the graph of the first from the graph of the second equation.

   (a) $y = 3(x + 4)^2$; $y = 3x^2$

   (b) $y = -2(x - 3)^2$; $y = -2x^2$

   (c) $y = -\frac{1}{2}(x + 1)^2$; $y = -\frac{1}{2}x^2$

   (d) $y = \frac{1}{3}(x + \frac{1}{2})^2$; $y = \frac{1}{3}x^2$

2. Set up a table of at least seven ordered pairs of the relation below, and then draw its graph.

$$y = 2(x + 2)^2$$

3. Complete the following table of ordered pairs for the equation

$$y = 2x^2 + 8x + 8.$$

| x | -5 | -4 | -3 | -2 | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
|---|---|---|---|---|---|---|---|---|
| y | 18 |  |  |  |  | 4.5 |  |  |

4. Draw the graph of the equation in Problem 3 and compare with the graph drawn in Problem 2.

5. Compare the location of each of the following graphs (without drawing the graph) with the location it would have if it were in the form $y = Ax^2$.

(a) $y = 3(x - 2)^2 - 4$
(c) $y = \frac{1}{2}(x - 2)^2 - 2$
(b) $y = -(x + 3)^2 + 1$
(d) $y = -2(x + 1)^2 + 2$

6. Find equations for the following parabolas.

(a) The graph of $y = x^2$
  (i) moved 5 units to the left;
  (ii) moved 2 units downward;
  (iii) moved 5 units to the left and 2 units downward.

(b) The graph of $y = -x^2$
  (i) moved 2 units to the left;
  (ii) moved 3 units upward;
  (iii) moved 2 units to the left and 3 units upward.

(c) The graph of $y = \frac{1}{3}x^2$ moved $\frac{1}{2}$ unit to the right and 1 unit downward.

(d) The graph of $y = \frac{1}{2}(x + 7)^2 - 4$ moved 7 units to the right and 4 units upward.

7. Set up a table of at least 7 ordered pairs for the relation below, and then draw its graph.

$$y = (x - 1)^2 - 4$$

8. Set up a table of at least 7 ordered pairs for the following relation, and draw its graph.

$$y = x^2 - 2x - 3$$

Compare this graph with that drawn for Problem 7.

## 4.12 Summary

In this chapter, we considered a number of experimental relations. In each experiment we considered possible variables which could affect the outcome, and arranged things to hold all but one of these fixed during the course of the experiment. From the data, we obtained a graphical (physical) model of the relation by drawing a smooth curve which seemed to give a best fit to the data points. We then considered various new relations between a mathematical model of the relation.

The experiments in this section gave rise to parabolic relations and led to some discussion of quadratic equations.

# Chapter 5

# ANALYSIS OF NONLINEAR FUNCTIONS

## 5.1 Introduction

Nonlinear functions and their graphs open a door to many exciting mathematical ideas. In this chapter you will investigate a few of these ideas through experiments.

The slope of a line is familiar to everyone. But what do we mean by the slope of a curve? Can such a slope be defined without confusion, and is it important? You can probably guess that the answers to these questions are going to be "yes". The concept of the slope of a curve is an extremely important one, and will be developed in this chapter. We will do this by making an analysis of a ball rolling down an inclined plane.

The continuing use of quadratic graphs may have left an impression that there are no other nonlinear curves. The simple lens, however, introduces the hyperbola -- a different curve with interesting new properties. Finally, the floating magnet will introduce a curve which defies simple analysis.

## 5.2 The Inclined Plane

In this experiment the motion of a ball down an inclined plane is to be studied. A ball rolling down a plane will move from side to side as well as down the incline. Since the distance the ball rolls is a necessary measurement, side motion would complicate the data. To prevent this lateral motion, we will use a V-shaped piece of aluminum as the "plane". This plane is inclined to the horizontal by a small angle and a billiard ball is used as the rolling body. The general arrangement of equipment is shown in Figure 1.

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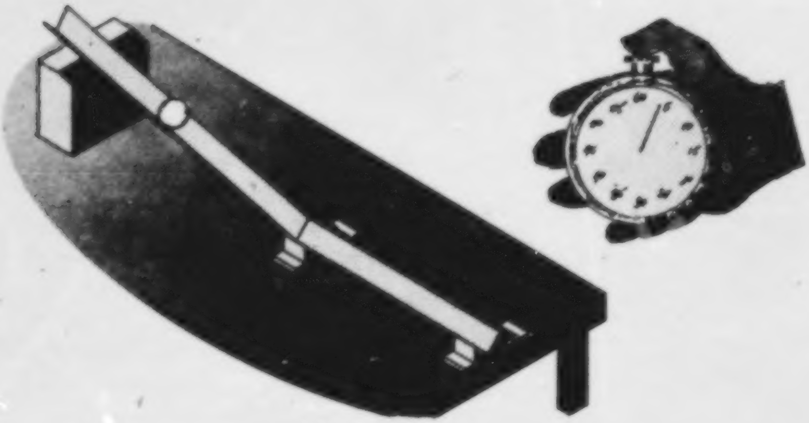


Figure 1

In the Falling Sphere Experiment, the sphere quickly reached a "terminal velocity" and from that point it fell with a constant speed. As the ball rolls down the plane, its speed will be constantly increasing. The speed of the ball at the end of a three-foot roll will be greater than its speed at the end of a roll of two feet. In this experiment we are going to measure the time it takes for the ball to travel different distances. We will use only one ball throughout the experiment, so any possible differences caused by balls of different sizes or weights will not concern us. The angle of inclination of the plane is one part of the experimental arrangement which has a great influence on the time taken for the ball to roll a given distance.

A ball rolling down a steep incline will cover a fixed distance in less time than a ball rolling down a slight incline. Set the plane at a small angle to the horizontal and keep it at this angle throughout the experiment. A small angle will "slow" the ball enough to make time measurements relatively easy.

In this experiment we are going to allow the ball to roll certain fixed distances and, using a stop watch, determine the time that it takes to roll these distances. We could release the ball from the top of the plane and determine how long it takes for the ball to reach a certain mark on the plane. A second method would be to release the ball at certain distances from the bottom of the incline and determine the time for the ball to reach the bottom of the plane. This second method has certain advantages. You will always know exactly where the distance interval ends.

The V-shaped piece of aluminum should be about 2.5 meters long.

Measuring from the bottom of the inclined portion of the plane, make marks on the plane corresponding to 15, 30, 50, 100, 150, 200 and 240 centimeters. Set the ball on the 15 centimeter mark and use the stop watch to determine the time taken for the ball to roll to the end of the incline. It is important to release the ball and start the watch at the same time. A convenient method is to place a finger on the top of the ball and hold the stop watch in your other hand. A few trials will enable you to release the ball at the same time as the stop watch is started.

The time taken for the ball to travel to the bottom of the plane will depend on the distance from the bottom of the plane. The distance measurements, therefore, form our domain, and the associated time intervals will be the range. Notice that this experimental procedure is the converse of that used in the Falling Sphere Experiment. In the Falling Sphere Experiment we picked certain time intervals (domain) and determined the distance traveled in that time (range). Repeat the procedure for each distance, and record in tabular form. See Table 1. Make three trials for each distance. Calculate and record the average time taken for each distance. To do this it is necessary to add the times of the three trials and then divide by three (the number of trials).

| Distance (cm) | Trial 1 Time (sec) | Trial 2 Time (sec) | Trial 3 Time (sec) | Average Time (sec) | Arbitrary Time (sec) | Calculated Distance (cm) |
|---|---|---|---|---|---|---|
| | | | | | | |

Table 1

From this data form ordered pairs of the form (distance, average time). Now select suitable scales for distance and time, and plot the ordered pairs. See Figure 2. Again, a physical argument allows us to construct the physical model by joining the plotted points in some manner. Every distance along the incline plane will have a time value associated with it. A smooth curve through or near the experimental points is a realistic physical model of the data.

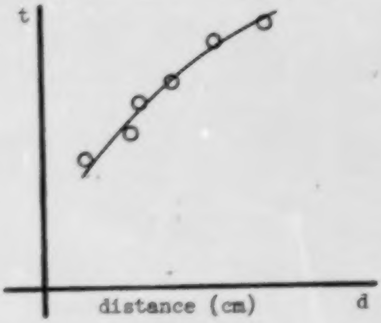


Figure 2

## 5.3 Analysis of the Experiment

In making an analysis of an experiment we attempt to relate the information to something from our past experience. In the Wick and Horizontal Metronome Experiments we met parabolic relations for the first time; we found a way to obtain a linear relation and then used our knowledge of linear relations to obtain an equation. In the Oscillating Spring Experiment, our first attempts at finding a linear relation met with failure. We then discovered that the converse of the relation had the same orientation as the curve found in the Horizontal Metronome Experiment. Once we relized this fact, we were able to relate the graph to something familiar and obtain a mathematical model of the experiment. The physical model of our present experiment, as shown in Figure 2, looks similar to the one found in the Oscillating Spring Experiment. Let us try to repeat the successful procedure used in the Oscillating Spring Experiment.

Form the converse of the distance-time relation. Interchange the domain and range such that the domain is now the set of time values, and the range the set of distances. Use these ordered pairs to plot a new graph. When graphed, the new figure will be similar to the curve of the Wick Experiment. We previously found that it was convenient to look for a linear relation between some power of the time (t) and the distance (d). When found, this gives us enough information in the proper form to directly write down a likely time-distance relation.

Square each of the time values and construct ordered pairs of the form $(t^2,d)$. Use the horizontal axis as the $t^2$ axis, and the vertical axis as the d axis. A line drawn through these points and extended will come very near the origin. This should be obvious since at zero time the ball will not have moved. Using the point (0,0) as the fixed point, draw the "best straight line" through or near the other points, as in Figure 3.

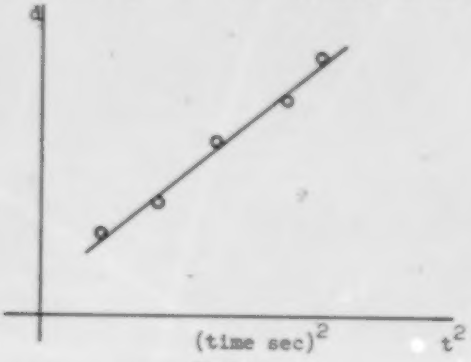


Figure 3

Compute the slope, m, of the line. We know that all lines which pass through the origin have an equation of the form $y = mx$. That is, the value of the range is equal to the slope times the corresponding value of the domain. Similarly, from Figure 3 the distance (range) is equal to the slope times the time squared (domain). This gives the equation

$$d = mt^2 .$$

We now must check this equation to see if it can be used as a mathematical model for the graph of the (t,d) relation. We can insert values of t in the formula $d = m \cdot t^2$ and calculate a value of d for each value of t. For ease in computation, select values of t which are integers. For example, if the time taken for the longest distance was 7 seconds you should calculate a value of d for every second from 1 through 7 seconds. Record these values in the next two columns of your prepared data sheet. Label the columns "arbitrary time (t) in seconds" and "calculated distance (d) in centimeters". Plot the ordered pairs formed by these two columns on the same sheet of coordinate paper as your original (t,d) curve. Connect the calculated points with a dashed line. The two curves should compare favorably.

Our equation can be used as a mathematical model to describe the behavior of a ball rolling down an inclined plane. There is one modification which should be made. As before, our equation was derived by use of a linear relation where the letter m has a special meaning. In our previous use of m it has denoted the slope of a line. Now our equation is not a line, and therefore m as the slope of a line would have no meaning.

Let us change the notation of our equation so the letter m does not occur. If we replace m with the letter A, we will not think of this as representing the "slope of the curve". Our final mathematical model is

$$d = At^2 .$$

## Exercise 1

1. Use the equation $d = At^2$. With your measured value of the coefficient A, calculate distance values that correspond to times of: 0, 1, 2, 3, 4, 5, 6, 7 seconds.
2. Draw a vertical line on a piece of graph paper to represent the inclined plane. Starting at the top, mark to scale the calculated positions of the ball along the inclined plane. Label these positions with the corresponding times.

3. On the drawing of the inclined plane in the exercise above, very carefully mark the position you think the ball will occupy at a time of 2.5 seconds. Using the equation, now calculate the position of the ball for this time. Compare this point with your estimated position.
4. Multiplying your value of "A" by four will form a new equation. With this equation calculate distance values for times of 0, 1, 2, 3 seconds.

## 5.4 Slope of a Curve at a Point

At this point we have a graph of our data and a physical and mathematical model which are abstractions of this data. There are many more aspects of the curve which are of interest to us. Mathematics, as the physical sciences do, sets up methods by which one curve or physical system can be compared with another.

Take a look at the graph of the parabola in Figure 4. In the region of the origin the curve is quite flat. That is, it is not rising very rapidly. Small time intervals along the horizontal axis correspond to very small changes in distance on the vertical axis. In this region it behaves similarly to the bottom of a mixing bowl. As you move out from the origin, the graph steepens and rises more rapidly for equal intervals along the horizontal axis. The same thing happens as you consider points farther out from the bottom of our mixing bowl. The problem before us is how to describe this behavior precisely.

Place your ruler to the right of the curve on your graph and select a shallow or small slope. Move your ruler parallel to itself until it just kisses the curve. See Figure 4. The ruler should just touch the curve and not cross it. The point of contact should be very close to a point. With the ruler in this position, draw a straight line and mark the point where it touches the curve. Now you have constructed a line, $\ell_1$, with a particular slope, and it touches the curve at only one point.

Construct a second line, $\ell_2$, with a much steeper slope. Use the same method as described above. Figure 4(c) is an illustration of the relative position of two possible lines.

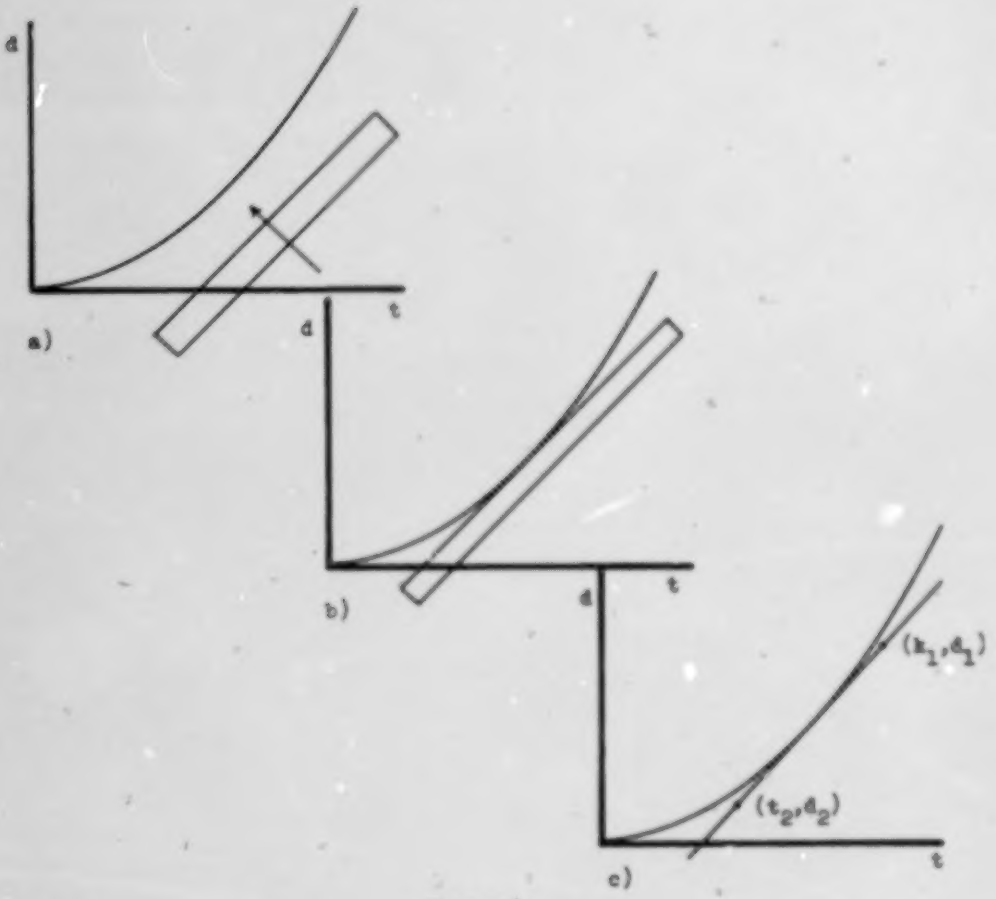


Figure 4

It is evident that when the steepness of the kissing line is small, the curve is very flat and rises slowly for intervals along the horizontal axis. When the curve is rapidly rising, the steepness of this line is large. We now have a quantitative way of describing the steepness of a curve. The steepness of any curve at a point can be given a number. Let us define the slope of a curve at a point as the slope of the straight line which just touches the curve at that point. A curve is twice as steep at one point as it is at another if the slope of the kissing line at the first point has twice the slope of the kissing line at the second point.

To firmly fix these ideas, take a point at approximately the 100 cm mark on the distance axis and locate the corresponding point on the curve. Measure the slope by drawing a line just kissing the curve at this point.

Make a few trials by angling your ruler before actually drawing the line. Compute the slope of the line. As stated above, this is the slope of the curve at the point where the line touches the curve. The slope of this curve has a particular physical significance. In this case, the slope is a distance (d) divided by a time (t). Another interpretation of distance/time is velocity. The question naturally arises as to what velocity does this slope measure. Consider the portion of the curve near the origin where the ball has covered a short distance and note the slope is small. Also, recall that near the origin the ball has a very low velocity. As you move out from the origin along the distance axis, the slope increases; that is, the velocity increases. Our observations have verified this. The greater the distance the ball travels, the greater the velocity. It is logical and also correct to interpret the slope at a point as proportional to the velocity the ball will have after traveling the distance d. In general, the slope of a distance-time graph at the point (d,t) is the velocity the object will have after traveling the distance d.

## Exercise 2

1. Carefully draw a graph of the parabola $y = \frac{1}{4}x^2$, using integrally spaced values of x from -6 to +6 inclusive. Graphically find the slope of the parabola at the points for which x equals 6, 4, 2, 0, -2, -4, -6.

2. The straight line is characterized by a constant slope whereas the quadratic has a continuously changing slope. It is possible to find the slope for many points on the curve, and hence, generate a new function which would consist of ordered pairs composed of slope and the elements from the domain.

   From the slopes found in Problem 1, form a set of ordered pairs (x, slope). On a sheet of graph paper, draw coordinate axes and plot this set of ordered pairs. What conclusions can you draw about this new function?

3. Compare the slope of the curve in Problem 2 with the coefficient of $x^2$ in Problem 1.

## 5.5 Experimental Measurement of the Slope

In the previous section we have defined the slope of a curve at a point to be the slope of the straight line which just kisses the curve at that point. The dimensions of this slope are the same as the dimensions of velocity. However, we have not proven that this slope can be interpreted as velocity.

The ball has a velocity at each and every point as it rolls down an inclined plane. If we could but measure this velocity by some experimental means, we could compare the result with the slope measurement taken from the time-distance graph. If the two are found to be the same, we can then say with conviction that the slope of the graph at any point is truly the velocity of the ball at that point.

All that remains is to find a way to measure the velocity of the ball experimentally. For this purpose a four-foot horizontal section of the aluminum angle is butted up against the end of the inclined plane, as shown in Figure 2. The two grooves should mesh as smoothly as possible. This smaller section of aluminum angle should be carefully leveled after placing it on two globs of modeling clay. The leveling can be accomplished easily by placing the ball on the track and seeing if it will roll one way or the other. The horizontal section of track provides a means for "tapping off" any velocity we choose. The ball rolls down the incline,increasing its velocity as it goes. When it rolls onto the horizontal track, the velocity no longer increases. It remains constant. The unchanging velocity of the ball, while on the horizontal section, will be exactly the same as the velocity the ball had the moment it left the incline. This velocity is computed from the measurement of the time needed to cover a set distance on the horizontal track. The value of the velocity is given by the quotient of the distance and time (velocity = distance/time).

This velocity can be adjusted by starting the ball at various positions up the incline. First, however, let us commit ourselves as to the velocity expected. Go back to your graph of the time-velocity relation and find the point corresponding to a distance of 150 cm. At this point, draw the kiss line. Measure the slope of this line and express it as a velocity in centimeters per second. This is the velocity the slope concept predicts for us after the ball has been allowed to roll 150 cm down the incline. This is the velocity we will measure experimentally.

Mark a length of 100 cm along the horizontal track starting from the end of the incline. This is the distance over which the motion of the ball will be timed. Release the ball from the 150 cm point on the incline, start the stop watch the moment the ball enters the horizontal section of track, and stop the watch at the moment the ball passes the 100 cm mark. Try this a few times before taking data. Now make three trials and record the measured time intervals. The velocity of the ball is now determined by dividing the distance traveled (100 cm) by the average of the three observed time intervals. Make this calculation and then compare this measured velocity in centimeters per second with the previously measured slope of the kiss line. Allowing for some experimental error, are these two figures the same? If these two figures are the same, we have proved our point. However, if these two figures are not the same, you should check both your kiss line and your measured velocity. The slope of the time-distance graph at a point is the velocity of the ball at that point!

It is worth noting that our procedure would enable us to directly measure the velocity of the ball after moving any desired distance down the inclined plane. This would enable us to compare the velocity to the slope of the line which kisses the curve at any point.

## 5.6 The Simple Lens

The use of a lens is most likely not new to you. Your science teachers may have used a lens when you studied vision, or in explaining how a camera works. You know that a lens will bring the rays of the sun to a focus.

If you mount a lens on a meter stick with a little modeling clay and "aim" at some distant object, you can find the image of this object on a white card on the other side of the lens. The image will be upside down and reversed left for right, but this need not bother us.

If you point the lens at some nearby object, you will find that the card will have to be moved to obtain a sharp focus. For distant objects, however, the image will always be found in about the same place.

Point the lens at a distant object outside of the classroom such as a building or a tree. (Be sure the window is open.) Move the cardboard screen until you have a "sharp" image of this distant object on the card. Measure the distance from the center of the lens to the screen. This distance is called the focal length of the lens and the position of the card is the focal point. Make three determinations of the focal length. Find the average value of the three trials and use this value as the value of the focal length. If you turn the lens around so the other side faces the object, the focal length will be the same. Thus, a lens has two focal points, one on each side of the lens, each of which is the same distance from the lens. See Figure 5.

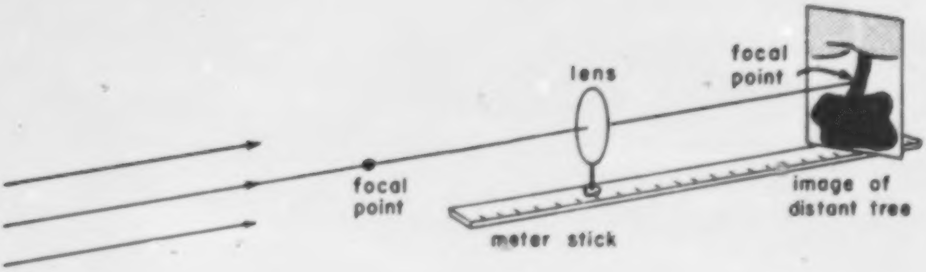


Figure 5

We have seen that for distant objects the image is always formed at the same distance from the lens. For short object distances, however, this is not the case. As we move objects closer to the lens, the image "moves" away from the lens. The relation between the position of our object and the corresponding position of the image formed by the lens will be the subject of our investigation.

We will need a brightly illuminated object for the experiment. Cut a small triangle in a piece of cardboard. Insert a pin into the base of the open triangle. This pin will be our "object". Darken the room somewhat during the experiment and place a flashlight directly behind the triangular hole to provide illumination. Obtain a piece of adding machine tape about two meters long. Fasten this tape to the floor and place the lens at the center of the tape. Try to arrange the lens so that its height is about the same as the height of the object. The experimental setup is shown in Figure 6.

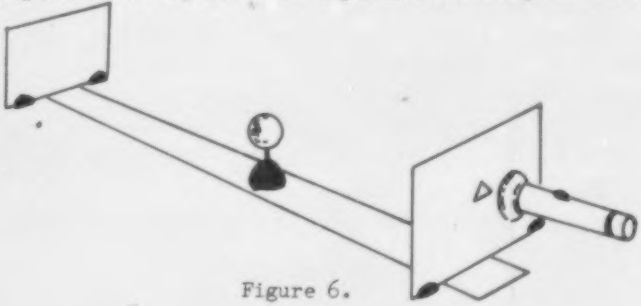


Figure 6.

To become familiar with the general behavior of the lens, first place the object so that its distance from the lens is about twice the focal length. Place the flashlight directly behind the object and turn off the other lights in the room. Move the screen until you have the image in sharp focus. As the object is moved towards the lens, the screen can be moved back to find a new position of sharp focus. Now move the object and flashlight so that the distance of the object from the lens is slightly less than the focal length. It is now impossible to obtain an image on the screen. We now know that our object distances must be greater than the focal length. When we place the object at about two focal lengths from the lens you should find the image position also about two focal lengths from the lens. As we move the object farther from the lens, the image moves closer until, for very large distances, the image is at the focal point. Thus, our object and image distances will always be greater than the focal length. Carefully measure from the lens to the focal point on each side of the lens, and make the two corresponding marks on the adding machine tape. These will be our two reference points.

We will measure distances from the focal points and not from the lens.

On the "object side" of the lens, use a meter stick and make a mark on the tape every centimeter from the focal point to the end of the tape. Repeat this process for the image side of the lens again starting from the focal point. Place the object on the last centimeter mark. Always remember to move the flashlight with the object so that you get about the same illumination each time. On the other side of the lens always move the screen until you find the point of "sharpest" focus. Make a two-column table; label the two columns "object distance (X)" and "image distance (X')". The symbol X is used instead of the letter O so you will not confuse this with the number zero. Be sure to measure the object and image distances from the focal points. The measurements are to be made as shown in Figure 7.

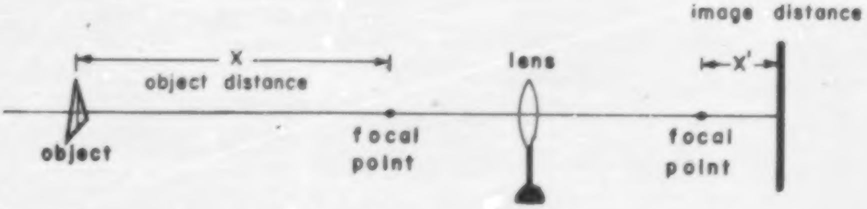


Figure 7.

After finding the location of the image when the object is at the last centimeter mark and recording this information in your table, move the object closer to the lens. The object should be moved two centimeters at a time, until the image is clearly out of focus. Move the screen to bring the image back into focus. Measure and record the new object and image distances. As you move the object closer to the lens, the screen will have to be moved away from the lens. For each reading continue to move the object (and the flashlight) two centimeters at a time until the image is definitely out of focus. Then move the screen until the image is back in sharp focus. Repeat this process for a number of trials until the screen is no longer on the tape.

Once we have collected the data, we will plot the object-image ordered pairs (X,X'), draw a physical model, and then attempt to find a mathematical model to represent and explain the relation. Notice that this threefold operation has been our plan throughout the text.

(a) Obtain data relation and graph.

(b) Construct physical model (best line or curve).

(c) Find mathematical model.

Set scales on the coordinate paper so that the graph will "fill" the paper. The image position depends upon the object position. The set of object positions is therefore the domain and is plotted along the horizontal axis. The set of associated image positions forms the range and is plotted along the vertical axis. You probably will have a graph something like that shown in Figure 10. Again we have the question of rilling the space between the points. As we move closer to the focal point of the lens, the image moved away from the other focal point. With every intermediate object distance there must be associated a new image distance. For object distances about three focal lengths from the lens or more, it may appear that you can move the object a few centimeters and still have the same image position. The image has, however, moved a small distance that is often difficult to detect visually. The

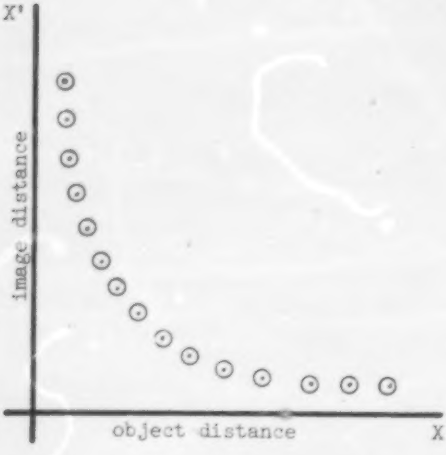


Figure 8

procedure for drawing a "best curve" seems fully justified in this case. This curve is our physical model of the object-image relation.

### Exercise 3

1. In the lens experiment what is the domain and what is the range?
2. Does the graph of the relation (Figure 8) represent a function? Why?
3. Would it be meaningful to pass a smooth curve through the plotted points? Why?
4. Discuss the possibility of extending the graph of the curve to very large or very small object distances.

## 5.7 The Lens Relation

Obviously, X and X' do not form a linear relation. As you recall, we were able to obtain an equation to represent parabolic relations by finding a linear relationship between some power of a number in the domain and the corresponding number in the range. For example, in the horizontal metronome relation we took ordered pairs of the form (d,T) and from these formed another relation with ordered pairs of the form $(d^2,T)$. This gave us a linear relation from which we were able to find an equation to represent our curve.

In this experiment the curve is not linear nor does it resemble the parabolic relations. We know, however, that as X decreases, X' increases; that is, as the object approaches the lens the image moves away. This type of behavior rules out forms like $(X^2,X')$. Why? Notice, however that as X decreases, a quantity such as $\frac{1}{X}$ increases. This means that as $\frac{1}{X}$ increases, X' will also increase. Although the relation (X,X') was not a linear relation, perhaps a set of ordered pairs of the form $(\frac{1}{X},X')$ will be. Select elements from the domain (X), form $(\frac{1}{X})$ values and then associate with these the appropriate elements from the range (X'). Enter the values of $\frac{1}{X}$ in a new column on your data sheet. Use a new sheet of coordinate paper and plot the relation formed by this new set of ordered pairs $(\frac{1}{X},X')$. The graph of these points should appear linear. Hence, $\frac{1}{X}$ and X' do form a linear relation. If this had not been so we might have attempted pairs such as $(\frac{1}{X^2},X')$, etc. Your new graph probably looks like that shown in Figure 9.

The best straight line through these points should come very close to the origin. For very large values of X, $\frac{1}{X}$ can be extremely small. For example, we could use the sun as an object. This would make X greater than

$10^{12}$ cm and $\frac{1}{X}$ practically zero. By our definition of focal point, the image of the sun would be formed at the focal point. Hence, when $\frac{1}{X}$ is practically zero, the image distance X' will also be about zero. From this line of reasoning we see that the line, when extended, should pass through the origin.

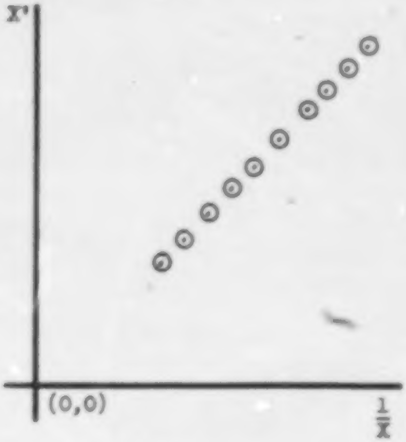


Figure 9

Now, using the origin as a starting point, draw your best straight line through the data points. This is our physical model of the experiment. Calculate the slope m of this line.

We know from our work with linear functions that all lines passing through the origin are of the form y = mx. Therefore, the equation of our line is

$$X' = m(\frac{1}{X}) = \frac{m}{X}.$$

We still must determine if this equation can be used as a mathematical model of our original curve. Using your experimental values of X, and the equation $X' = \frac{m}{X}$, compute corresponding values for X'. The calculated value for m should be used in the equation. Enter the calculated values of X' in a new column of your data table. Plot the new ordered pairs (X,X' calc.) on the same sheet of coordinate paper as your experimental points. Use small solid circles for the points and connect them with a "dashed" curve. Compare the calculated and experimental curves. The two curves should compare favorably We can now say that

$$X' = m(\frac{1}{X}) = \frac{m}{X}$$

can be used as a mathematical model of our experiment.

Although both the domain and range of our data function were somewhat limited, we have every reason to believe that the above equation is valid for all values of X and X' where both are greater than zero. This conjecture, of course, should be tested by further experimentation.

It is important to realize that the symbol "m" in the equation $X' = \frac{m}{X}$ is not the slope of the graph of the (X,X') relation. It is, on the other hand, the slope of the $(\frac{1}{X},X')$ relation. For this reason, it is best to replace the symbol "m" by some other symbol that indicates a constant value. But what constant is it? You have obtained the numerical value of this constant, and it is interesting at this time to compare it to the square of the focal length (f) of the lens that was used. Allowing for some experimental errors, you should find that $m = f^2$. This is the case, and we can now write our lens relation in the final form

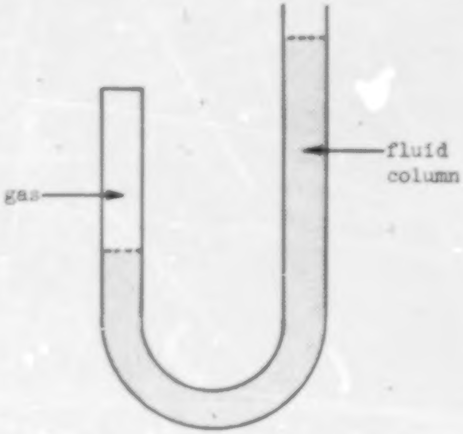
$$X' = \frac{f^2}{X}.$$

Our equation now suggests that an extremely important generalization of our lens relation can be made. Perhaps this equation can be used to represent the location of the object and image for any lens that we may wish to use. This turns out to be the actual case, as has been verified in many experiments in the past.

## Exercise 4

1. The following table contains data taken from an experiment with gases.

| Pressure $lb/in^2$ | Volume $cm^3$ |
|---|---|
| 4 | 169 |
| 5 | 135 |
| 10 | 68 |
| 12 | 56 |
| 15 | 45 |
| 18 | 38 |
| 20 | 34 |
| 25 | 25 |
| 30 | 23 |
| 35 | 19 |



By raising and lowering the fluid column different pressures can be exerted on the gas contained in the left portion of the tube. As the fluid column is raised, the pressure is increased and the gas volume decreases.

(a) Which elements of the table are the domain and which are the range?

(b) On a coordinate plane, plot the ordered pairs from the table and construct a physical model.

(c) Form a new relation $(\frac{1}{p}, V)$ and plot these new ordered pairs.

(d) Using this information, find the mathematical model which best represents the data.

## 5.8 The Reciprocal Function

In this experiment we have obtained the relation

$$X' = \frac{f^2}{X}.$$

This is a particular example of a more general relation

$$y = \frac{k}{x},$$

where k is some constant.

You will recall that we have studied relations of the form $y = mx$. We call these relations "linear" because they graphed as a straight line. We also studied relations of the form $y = Ax^2$, which were called quadratic relations. "Quadratic" comes from the Latin word quadratus, meaning squared. Now we are concerned with a relation of the form $y = k(\frac{1}{x})$. In this case, y varies as the reciprocal of x. For this reason, let us call this a reciprocal relation. The graph of $y = \frac{k}{x}$ reveals the basic pattern of this relation. For extremely large values in the domain, the corresponding values in the range are very close to zero. For extremely small values in the domain, the corresponding values in the range are extremely large. This reciprocal relation is clearly a function, for to every element in the domain of the function, there corresponds one element in the range. We can also see this graphically. No vertical line cuts the curve in more than one place.

Let us consider the graph of the relation $y = \frac{k}{x}$ for negative values of x in spite of the fact that negative values of X in our experiment apparently have no physical significance. Now graph $y = \frac{k}{x}$ for all possible values of x. From our experiment, $k = f^2$, so k is greater than zero. Since k is positive, x and y must both be positive or both negative.

Thus, when the domain of the function $y = \frac{k}{x}$ is extended to include all real numbers $\neq 0$, the graph we obtain is shown in Figure 10. Why is zero excluded from the domain and range of this function?

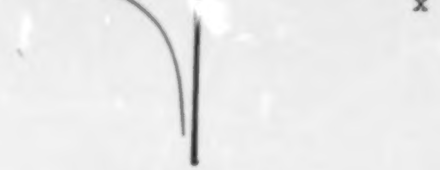


Figure 10

This more complete relation is a function because we still have one element in the range of the relation which corresponds to each element in the domain. Note that the domain excludes only the single value zero. This reciprocal function is so important that its graph has been given the special name "hyperbola". All hyperbolas have two portions, as shown above. Hyperbolas are very closely related to the parabolas we encountered in Chapter 4.

All relations have converse relations, and we should inquire as to the converse of a reciprocal relation. When we interchange the domain and range of this relation, we notice a curious thing. When we form the converse, each ordered pair that we obtain is seen to be the same as one of the ordered pairs of the relation itself. (Prove this to your own satisfaction.) This means that the graph of the converse reciprocal relation is identical to the graph of the relation. This same conclusion could have been found algebraically by finding that $x = \frac{k}{y}$ .

Without going very far into a more complete physical analysis of a lens, let it be said only that negative object-values and negative image-values actually have as much significance as positive values of these same quantities. A negative X would arise for an object placed at any position to the right of the left-hand focal point. (The light rays are always considered to move from left to right.) Similarly, a negative X'is an image distance measured to the left from the right-hand focal point. The images that are obtained in these situations are not the kind that can be projected upon a screen. They can, however, sometimes be seen by looking directly into the lens.

## Exercise 5

1. Does the range of the function $X' = \frac{f^2}{X}$ include the value of $X' = 0$ ? Explain.

2. Does the simple lens equation $X' = \frac{f^2}{X}$ , with the range and domain restricted to the values that can be obtained experimentally, represent a function if X and X' are interchanged? Why?

3. The focal length of the lens found in many cameras is 50 cm. Calculate X' in centimeters for an object at a distance (X) of 1 meter; 10 meters; $1.5 \times 10^8$ meters (the distance to the moon); and $5.8 \times 10^{10}$ meters (the distance to the sun).

4. I. $y = \frac{10}{x - 2}$

II. $y - 5 = \frac{10}{x - 2}$

III. $y = \frac{-10}{x - 2}$

IV. $y - 5 = \frac{-10}{2 - x}$

V. $y = \frac{10}{|x - 2|}$

For each of the relations above,

(a) For what value of x will the denominator become zero?

(b) Is it possible for x to be equal to zero?

(c) Find the value of y which corresponds to the following values of x:

$$\{-8, -3, 0, 1, 3, 4, 7\}$$

(d) Using the values just found, form ordered pairs of the form (x,y) and plot on the coordinate plane.

(e) Join the points with a smooth curve. Remember that there will be one number (part a) which is not in the domain of the relation.

(f) If there any number which is not in the range of this relation? If so, what is it?

## 5.9 Translation of Axes

In Chapter 3, the discussion of linear equations led us to an investigation of the translation of axes. This translation was performed in two directions, both horizontally and vertically.

Further in the text a translation was also performed during the discussion of the parabola. In this case it was the curve itself which was translated.

It is now advantageous for us to translate the axes in the case of the hyperbola. Referring to Figure 7, you will recall that both object and image measurements were made from the focal points. The values of these

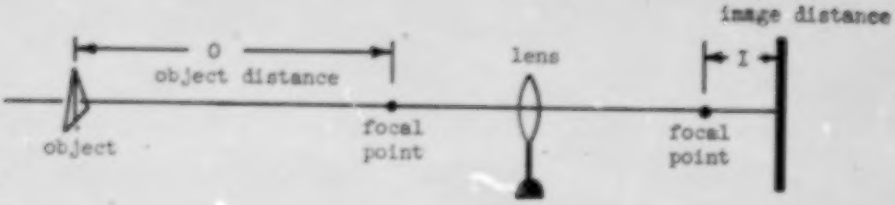


Figure 7

measurements were used in obtaining the ordered pairs from which the graph of the hyperbola in Figure 10 was drawn.

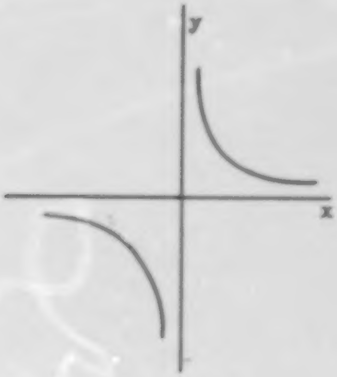


Figure 10

When the lens experiment was first performed, you measured the object and image distances from the focal points. You may have felt that this was not a natural point from which to start. A more logical starting point for making measurements would be the lens.

If it is desirable to make measurements from the lens rather than the focal points, this can be analyzed graphically by a translation from the graph of Figure 10. In the original graph, the origin was, in effect, the focal point. We now wish to have the origin represent the lens position. This requires a graphical translation.

If you recall, in our original data, the positive object distances were measured to the left. The origin is now being moved from the value of the object focal point to the lens position which is a movement to the right. Therefore, the translation is taking place in the negative direction. The distance from the lens position to the focal point is "f". From this it follows that the translation would have to be made by an amount f both downward and to the left. That is, we are shifting the axes in negative directions by an amount f. This shift is achieved by adding an amount -f to the object values and image values. The equation becomes

$$X' + (-f) = \frac{f^2}{X + (-f)} \text{ .}$$

We have added the horizontal translation (-f) and the vertical translation (-f) to the variables. Now, multiplying the equation by X + (-f) we find

$$(X - f)\ (X - f) = f^2 \ .$$

Remove the parentheses by applying the distributive property,

$$XX' - Xf - X'f + f^2 = f^2 \ .$$

Subtracting $f^2$ from each side of the equation,

$$XX' - Xf - X'f = 0 \ .$$

Rearranging,

$$XX' = Xf + X'f \ .$$

Multiplying the equation by $\frac{1}{X'Xf}$ , we have

$$\frac{1}{f} = \frac{1}{X'} + \frac{1}{X} \ .$$

This is precisely the equation for which we are looking. The quantities X and X' are object and image distances measured from the lens.

Now take a sheet of frosted acetate with coordinate axes. Place it over your graph obtained from the lens data. Translate the axes an amount -f in both directions. The curve then appearing on the frosted acetate is a representation of the translated curve.

We may conclude that the ability to translate coordinate axes is a technique that is extremely valuable. In this case it has erased the apparent physical difference between the two ways to measure the position of object and image. These two position descriptions change the mathematical description of our graph, but do not change the shape or relative position of the two portions of the hyperbola.

## Exercise 6

1. Start with the equation $\frac{1}{X} + \frac{1}{X'} = \frac{1}{f}$ whose significance is described in the text. Algebraically translate the axes to the right and upward by the amount f in each direction.

   (Hint: form the equation $\frac{1}{X + f} = \frac{1}{f} - \frac{1}{X' + f}$ and simplify.

2. Algebraically solve the equation $\frac{1}{X} + \frac{1}{X'} = \frac{1}{f}$ for X' .

3. $y = \frac{3x}{x - 3}$ is a hyperbola in the form found in Problem 2. By how much and in what directions would one have to translate the axes to put it in the form $y = \frac{k}{X}$ ?

4. Translate the axes used to describe the parabola $y = x^2 - 4x + 4$ so that the vertex of the parabola lies at the origin. By what amounts and in what directions did you translate the axes?

## 5.10 Curve Sketching

In our experiment with the simple lens we used the two focal points of the lens as points of reference for measuring the location of the object and image. In so doing we found the function $X' = \frac{f^2}{X}$ . We remarked in the previous sections that it may be awkward to measure distances from imaginary points that could neither be seen nor touched. We then elected to locate both the object and image with respect to the position of the lens. These two quantities are shown in Figure 11.

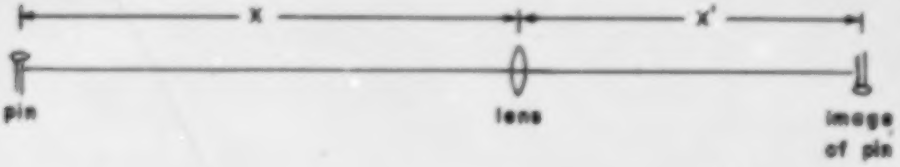


Figure 11

The relation between X and X' is given by

$$\frac{1}{X} + \frac{1}{X'} = \frac{1}{f} ,$$

where the constant "f" is the focal length of the lens. In Exercise 6, Problem 2, you solved the equation $\frac{1}{X'} + \frac{1}{X} = \frac{1}{f}$ for X'. The answer obtained should be

$$X' = \frac{Xf}{X - f} .$$

This equation expresses a relation and provides us with an excellent opportunity to perform an exercise in "curve sketching". The language "curve sketching" refers to a rough sketch of a curve that is made after observing a few important features of the equation. Few, if any, exact points need to be obtained to make the sketch. The general idea is to make the sketch just to see "how things go", and not to obtain exact representation of the graph.

To sketch the equation $X' = \frac{Xf}{X - f}$, we start with a sheet of coordinate paper upon which we draw the horizontal (X) axis and the vertical (X') axis. The succeeding steps taken to sketch this equation are listed below.

(1) Let us consider the above equation when the value of X is large and positive. Because f is relatively small with respect to X, the denominator, X - f, will remain practically the same as X itself. If this is the case, the entire fraction, and therefore X', will have a value only slightly greater than f. For example, use f = 6 and X = 1000. Then

$$X' = \frac{(1000)(6)}{1000 - 6} = \frac{6000}{994} = 6.04 .$$

Therefore, for large positive values of X the graph of the curve will stay close to the vertical coordinate X' = f.

(2) As the value of X becomes smaller positively, the difference between X and f becomes smaller, and therefore the value of the fraction becomes larger. For example, again use f = 6, but now have X = 100. Therefore, $X' = \frac{(100)(6)}{100 - 6} = \frac{600}{94} = 6.4$. Using f = 6 and X = 10, we find that $X' = \frac{(10)(6)}{10 - 6} = \frac{60}{4} = 15$. We see that the X' values increase very rapidly as the X value comes closer to f, and the curve becomes very steep.

(3) When X = f, the denominator of our equation becomes zero. Since such an expression is undefined, we cannot graph this point.

Using these three steps we can sketch a portion of our curve. Since both the X and X' discussed thus far are positive, the graph of the curve is confined to the first quadrant. See Figure 12.

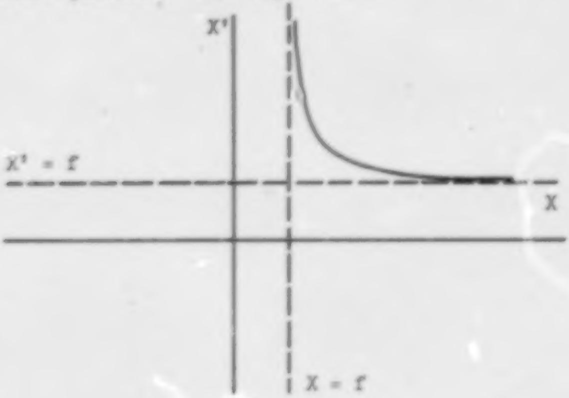


Figure 12

The graph of the curve approaches the value of f in both the horizontal and vertical directions. Therefore, it is helpful to draw the lines X' = f and X = f on the graph.

(4) Now let us consider the equation when X is less than f, but still positive. The denominator of the equation, X - f, would then be negative. The numerator is still positive. Therefore, the value of the fraction and, consequently, of X', is negative. For X values only slightly smaller than f, the denominator becomes a very small number. Therefore, the fraction itself becomes very large and is negative.

(5) When X equals zero then the value of Xf is zero. The denominator is not zero. Since the numerator is zero, the value of the fraction is zero. Therefore, the graph of the curve passes through the origin.

(6) As X becomes negative, the value of the numerator becomes negative. But the value of the denominator also becomes negative. Therefore, the value of the fraction is again positive.

(7) As X becomes very large and is still negative, the value of the denominator changes very little. The value of f is very small in relation to the large value of X, and the difference in the denominator remains very close to X. As we determined in (1), the value of the fraction approaches f as the value of X increases negatively. Since both the numerator and denominator are negative, the value of the fraction remains positive.

Using the last four steps, we can sketch the portion of the curve shown in Figure 13.

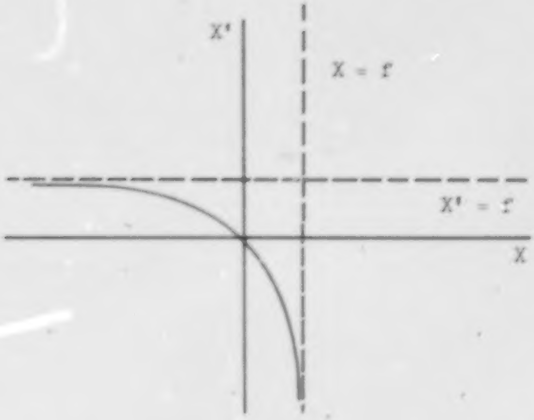


Figure 13

To sketch the graph of the curve for the equation $X' = \frac{Xf}{X - f}$ we must combine the conclusions we have reached in the seven steps above. Figure 14 illustrates the sketch of the graph for the given equation.

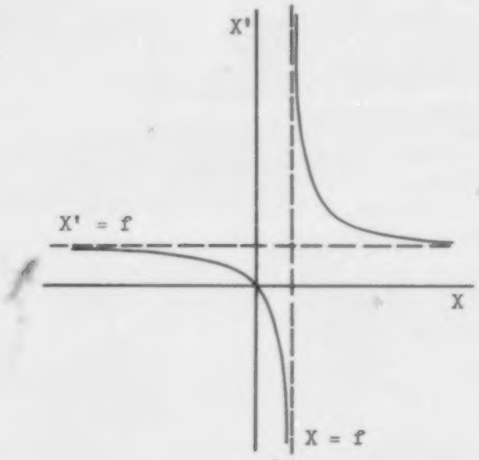


Figure 14

Go back over the seven features once more, and check each against the sketch. You should practice using sketching procedures such as this. With a little practice, sketches of most curves are easily drawn.

Exercise 7

Sketch the following relations for all possible values of x:

1. $y = \frac{6}{x + 3}$
2. $y = \frac{x}{x - 2}$
3. $y = x(x - 2)$
4. $y = x^2 - 2x + 1$
5. $y = 2(x + 1)$

## 5.11 The Floating Magnet

Perhaps nothing is more fascinating than a magnet. Two magnets may attract or repel one another. Magnets will attract tacks, paper clips, nails, and any other object that contains iron. Iron filings placed on a card over a magnet will form a beautiful pattern. Very strong magnets can be made by winding a coil of wire around a nail and attaching the ends of the wire to a small battery.

Magnetic phenomena are, however, extremely difficult to analyze. Magnets are so nice--but what can one really do with them? The experiment that follows represents one of the few experiments with magnetic phenomena that provide a real opportunity for mathematical analysis.

The magnets that will be used are small circular ceramic magnets about an inch in diameter that have holes in their centers. Four of these, each $\frac{1}{8}$-inch thick, are needed, or two that are $\frac{1}{4}$-inch thick. A knitting needle, paper clip, centimeter ruler, and a set of standard masses will also be required. The experimental arrangement is illustrated in Figure 15. The top two magnets are repelled strongly away from the lower two. They seem to float in midair without visible means of support. The knitting needle (cut off to a suitable length) passes freely through the holes in all four magnets and through the hole in the mounting board that is used for support of the entire setup.

Because the upper magnets cannot slide off the capped end of the knitting needle, the needle and upper magnets move together as a unit. The lower magnets simply rest upon the meter stick. The upper magnets, together with the knitting needle, are free to bounce up and down with the slightest push.

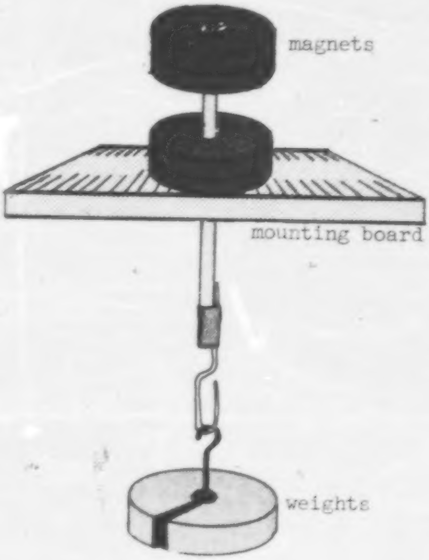


Figure 15

We wish to investigate the manner in which the separation distance between the magnets decreases as the load suspended from the knitting needle is increased. In this experiment, the selection of the two physical quantities of interest is rather clear-cut.

We will select a load ($\ell$), and corresponding to this load there will be a separation distance between the magnets (s). The load values are the domain of the function while the separation distances are its range.

In performing the experiment, it is perhaps best not to attempt a direct measurement of the separation distance. The magnets may tilt slightly on the needle one way or another. Instead, the distance (d) between the cut end of the knitting needle and the under side of the mounting board should be measured. This measurement should be read to 0.1 mm on your scale by estimating tenths between adjacent divisions. The distance (d) and the separation distance (s) are shown in Figure 16.

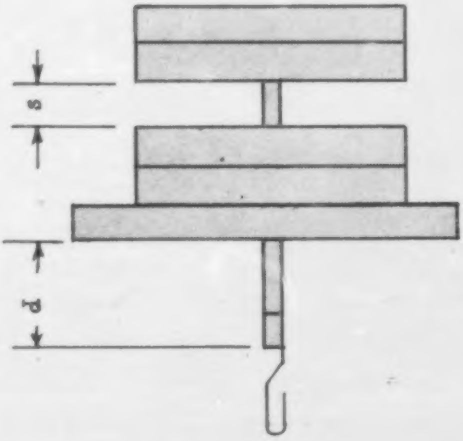


Figure 16

A load of about 160 grams will reduce the separation distance between the magnets to less than 1 mm, so if we load the knitting needle plunger in 20-gram steps, we will obtain about nine readings. Label the first column of your data sheet "load ($\ell$) in grams" and the second column "distance (d) in mm". (See Table 2.) Be sure to record the value of d when only the mass of the needle itself is applied to the upper magnets. When taking these readings, tap the needle gently to make sure that the friction between the needle and the holes through which it passes does not influence the results.

We must now change the measurements of d into the corresponding values for s, the separation distance. This is done easily by forcing the magnets together and finding the corresponding value of d. Call it $d_0$. The required values of s are then obtained from $s = d_0 - d$. Be sure to convince yourself that this equation is the correct one to use. Place the s-values found for each load in column 3 of your data table.

| load ($\ell$) in grams | distance (d) in mm | s values | $\frac{1}{s}$ values | s values 2 mm spacing | calculated $\ell$ values |
|---|---|---|---|---|---|
| | | | | | |

Table 2

The desired data function is now shown in columns 1 and 3 of your table.

The values of the load $\ell$ are the domain of the function, and determine the scale of the horizontal axis. The corresponding values of the separation distance s determine the scale along the vertical axis. Plot your $(\ell, s)$ pairs and draw your "best curve" through or near these points. This best curve is a physical model which assumes that for any intermediate value of the load, a corresponding value of the separation distance would have been found. Your graph of the physical model for the "floating" magnet should now look something like the graph shown in Figure 17.

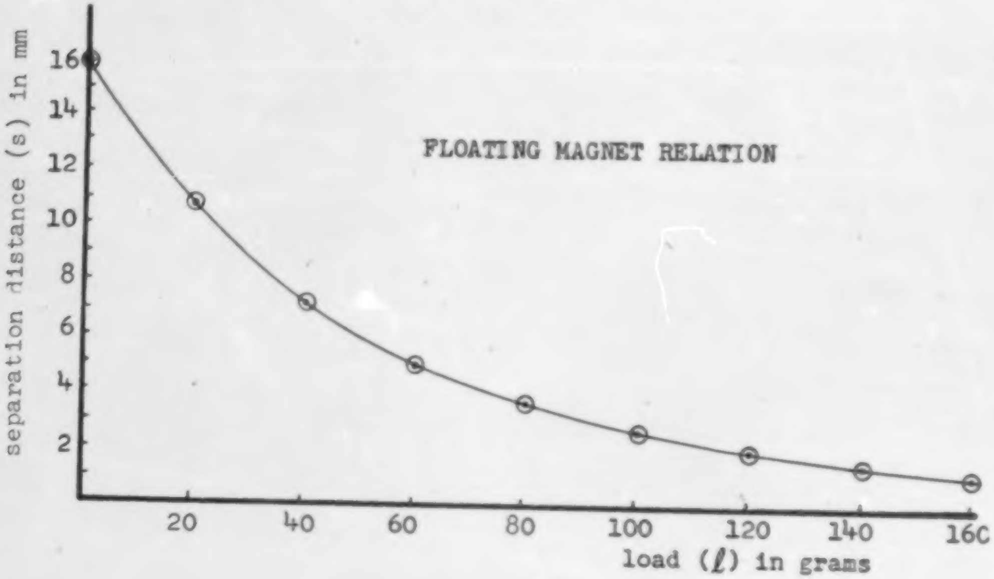
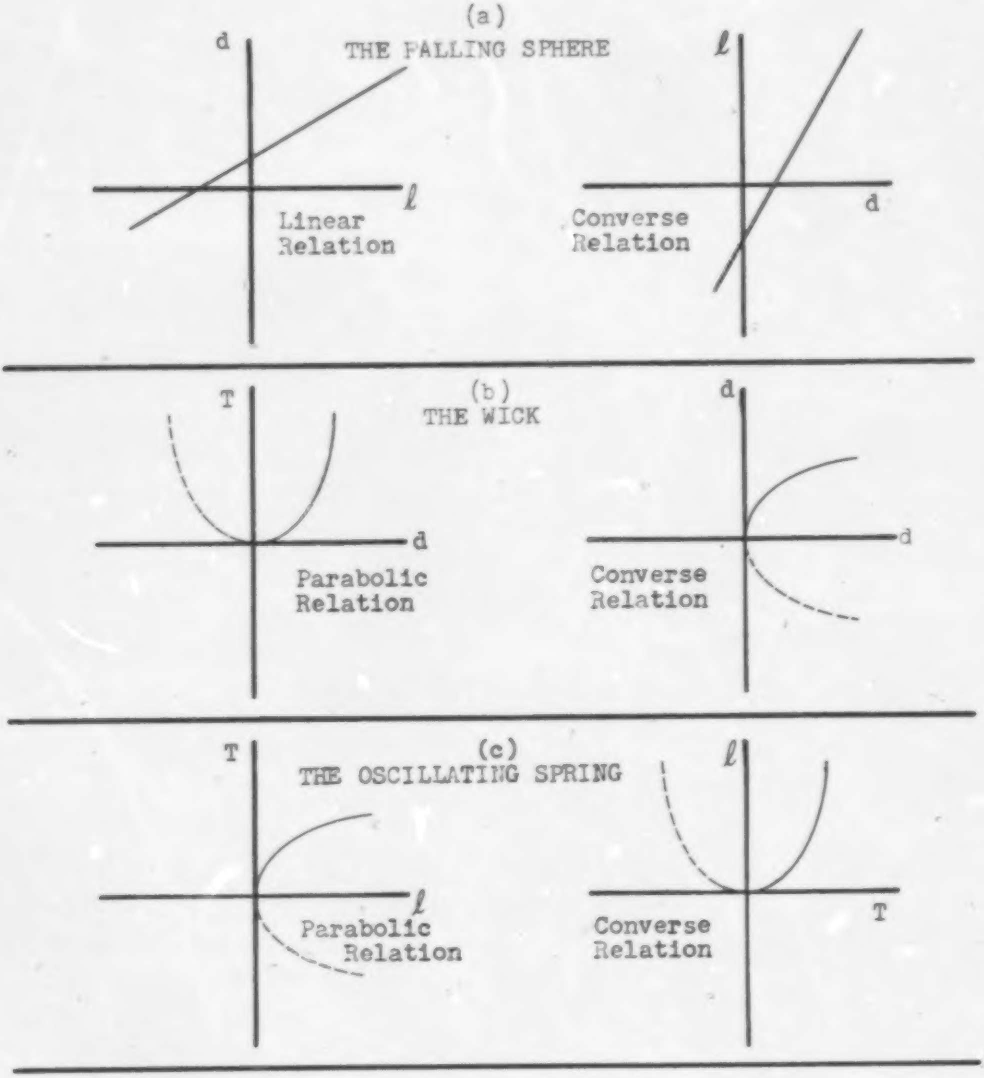


Figure 17

## 5.12 Search for a Mathematical Model

Your curve of the kind shown in Figure 17 is a representation of the results of the experiment, and we must now find a mathematical model (an equation) which describes this curve. At this point it might be to our advantage to look back at all the kinds of graphs we have encountered. One of these might well be the one we are looking for. Figure 18 on the next page shows the eight graphs studied in Chapters 1, 2, 3 and 4. We might call it a "Gallery of Graphs".

Figure 18(a) shows the linear relation obtained in the Falling Sphere Experiment. The converse relation is also linear. This relation was also encountered in the Number Generator Experiment. The relation shown in Figure 18(b) was obtained in the Wick and Horizontal Metronome Experiment. The parabolic relation was found with the Oscillating Spring and shown in Figure 18(c). This was found to be the same as the converse of the Horizontal Metronome and Wick relation. The reciprocal relation for the Simple Lens, Figure 18(d), is identical to its own converse.


(a)
THE FALLING SPHERE
d
ℓ
Linear Relation
ℓ
d
Converse Relation
(b)
THE WICK
T
d
Parabolic Relation
d
d
Converse Relation
(c)
THE OSCILLATING SPRING
T
ℓ
Parabolic Relation
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Converse Relation

## (d)

## THE SIMPLE LENS

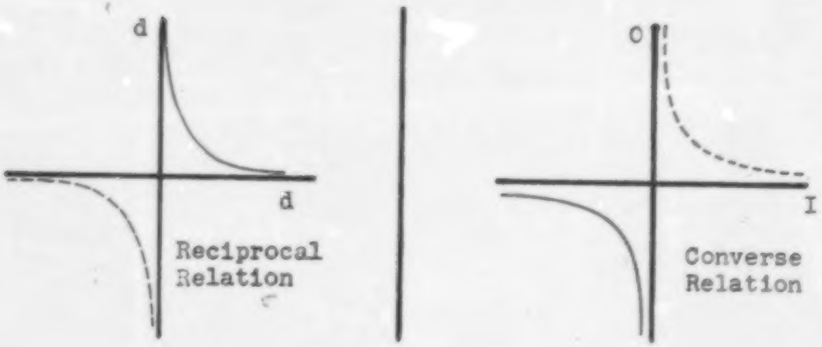

d
d
Reciprocal Relation
O
I
Converse Relation

Figure 18

We hope to find a graph in the gallery that has a shape similar to the shape we have already obtained for the floating magnet relation. Perhaps we may use only a portion of one of these graphs to describe the magnet relation. As we scan the "Rogue's" gallery, we see three possibilities:

(1) the dashed portion of the parabolic relation at the left in Figure 18(b);

(2) the dashed portion of the parabolic relation at the left in Figure 18(c);

(3) the solid curve portion of the reciprocal relation shown in Figure 18(d).

Let us examine these possibilities one by one.

If we were to use the left-hand portion of the upright parabola in Figure 18(b), we would have to translate the axes to the left so that this part of the parabola would appear in the first quadrant. See Figure 19(a). It must appear in the first quadrant, for that is the location of our floating magnet relation. Having translated the axes in this way, however, we find that the parabola shows one separation distance between the magnets for two different loads, as shown by the dotted line. This does not represent the physical situation and therefore this parabolic model cannot be used.

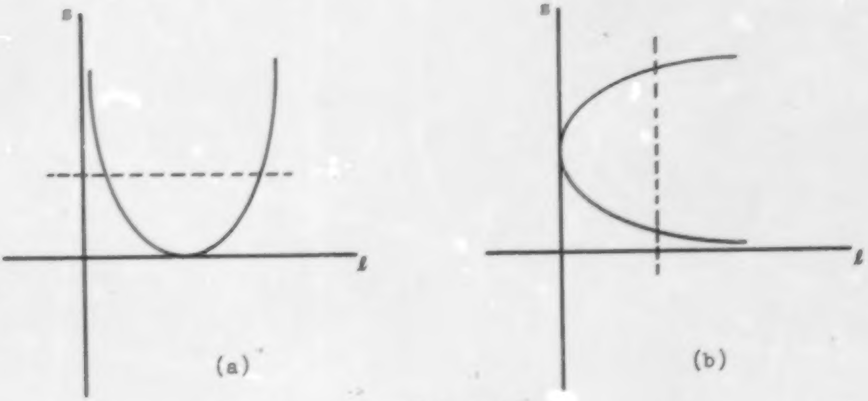


Figure 19

If we were to attempt to use the lower half of the parabola on its side, Figure 18(c), a similar situation would confront us. We would now have to translate the axes downward to place the dashed portion in the first quadrant, as in Figure 19(b). Now, however, the parabolic model would predict two different separations of the magnets for one load, as shown by the dotted line. Again, this is a physically impossible situation and the model must be discarded.

Suppose in the two cases above, we try to solve our problem by "throwing away" the half of the parabola we do not want. Then, in each case there would be an artificial limitation. In one case the domain of loads would be limited, and in the other case, the range of separation would be limited. Therefore, we must reject the possibility of half parabolas, for in both cases the limitations do not correspond to the physical situation.

No objections can be raised in the case of the reciprocal relation of Figure 18(d). The graph needs no translation to be similar to the floating magnet graph. It also does not indicate multiple loads for a single separation, or multiple separations for a single load. It is, therefore, the one we will employ in our attempt to describe the floating magnet relation.

## 5.13 The Reciprocal Relation

The function we have obtained consists of the ordered pairs ($\ell$,s) that are on the "best curve" we have drawn through the experimental points. We have now decided to represent this curve by a reciprocal relation. As you will recall from our study of the Simple Lens, the graph of the converse relation is a curve which is identical to the graph of the reciprocal relation itself. For the magnet relation, this means that we could follow either of two procedures. We could form a new domain consisting of $\frac{1}{\ell}$ -values and plot these against the corresponding s-values in the range, or we could use the converse magnet relation consisting of the ordered pairs (s,$\ell$), form a new domain consisting of $\frac{1}{s}$-values, and plot these against the corresponding $\ell$-values in the range.

Faced with these two possibilities, we must make a choice. If we remember that the very first value of the load ($\ell$) that we placed in our table was 0, we can see immediately that the corresponding value of $\frac{1}{\ell}$ is not defined. No similar difficulty arises for $\frac{1}{s}$ because s did not assume the value 0. Let us hope, then, that a graph of pairs ($\frac{1}{s}$,$\ell$) will yield a straight line. If it does, we will have found the reciprocal relation we are seeking.

Before going farther, however, we should graph the converse magnet relation which consists of the ordered pairs (s,$\ell$). Replot your data points and draw a new "best curve" on a sheet of coordinate paper to obtain a graph of the converse relation. Your new graph should be similar to the one shown in Figure 20.

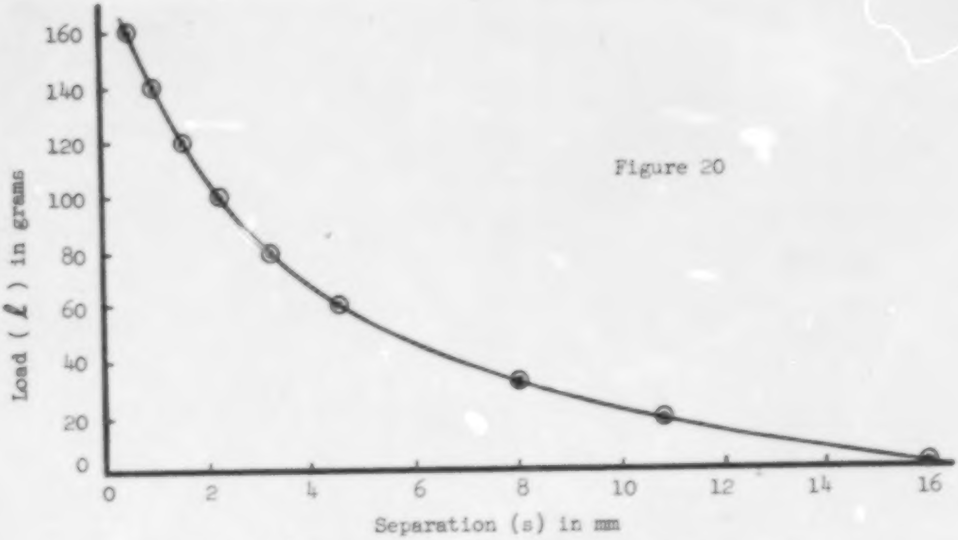
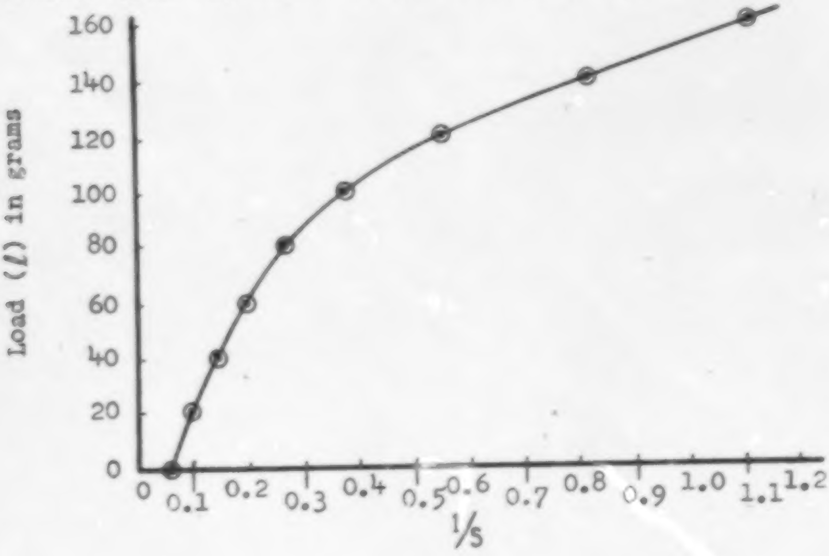


Figure 20

Now we must tabulate $\frac{1}{s}$ values corresponding to each value of s. Place these in column 4 of your data table. Now graph the $(\frac{1}{s}, \ell)$ relation using the horizontal axis for $\frac{1}{s}$ values and the vertical axis for the values. This is the relation we hope is a linear one, for if it is, the relation between s and $\ell$ will then be a reciprocal relation.

Draw a "best curve" through these points. When this is done, your graph should appear very similar to the one shown in Figure 21.



This graph is disappointing. It does not seem to be a line by any stretch of one's imagination. This means also that the relation between s and [illegible] is not a reciprocal relation.

But look more closely at your graph. Although the entire graph is most certainly not a line, the first four points (small loads, large separations and low $\frac{1}{s}$ values) do line up fairly well. This restricted part of the graph shows a linear behavior. But when the loads become too large and the separations too small, the graph curves off to some new sort of relation.

Let us draw a "best straight line" through these first four points and find the corresponding mathematical relation. See Figure 21. It is true the mathematical equation will not describe the remainder of the graph, but at least it should provide an accurate description of the behavior of the floating magnets for small load values.

The equation representing this "best straight line" will be of the form

$$\ell = m\left(\frac{1}{s} - c\right).$$

This is the familiar point-slope equation. In this firm it is relating $\frac{1}{s}$ to $\ell$ .

The values of the constants "m" and "c" are found from the graph of the line. The constant c is not equal to zero because at "zero load" we still have the loading of the upper magnets and needle which influence the separation distance. This, then, is the mathematical model we have been seeking. We must recognize that this model does not pretend to describe the entire behavior of the floating magnet, but only that part of its behavior that corresponds to small loads. Notice also that we have obtained a relation that is the converse of the experimental relation. The above relation predicts the values of the load for certain fixed values of the separation distance. In the experiment, the separation distance was determined by the load.

One final step will make our analysis complete. We should now use this equation to obtain pairs (s, $\ell$ ) to compare directly with the results of our experiment. We are sure that these calculated points will not match the experimental curve for large loads and small magnet separations. In spite of this, we will calculate to see how good the mathematical model is for large separation distances and how poor it is for small separation distances.

In column 5 of your data table select s-values spaced every 2 mm over the entire range of the original function. Place the calculated values of the load ($\ell$) found from your equation in column 6. Plot these calculated points on the same sheet of coordinate paper used to display the converse relation (as in Figure 20). Draw a dashed line through these points to distinguish the graph of the mathematical model from the graph of the experimental results.

How do the two graphs compare? You should have obtained a result similar to the one indicated in Figure 22.

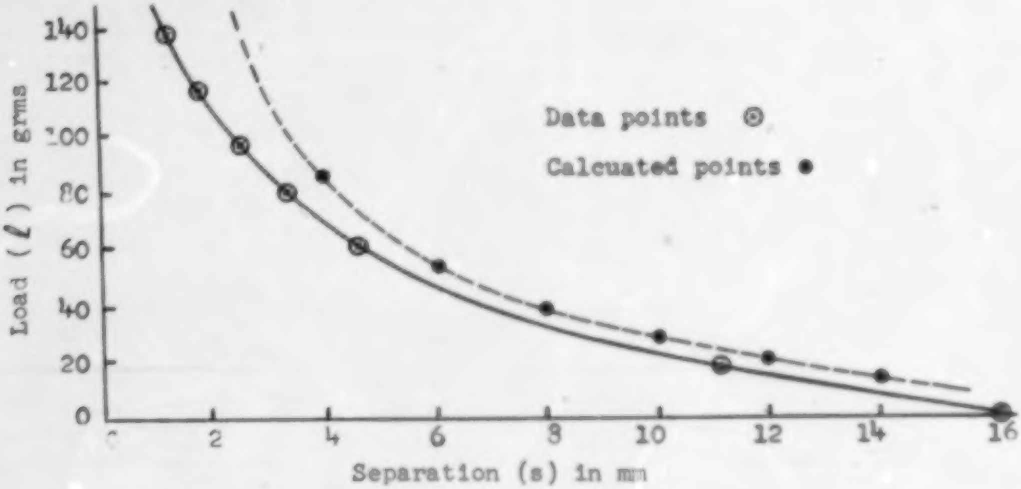


Figure 22

We see that the curve calculated from the mathematical model represents the behavior of the floating magnet for loads that are sufficiently small. Predictions from this equation for larger loads, however, would not agree with the actual behavior of the magnets.

The restriction that we have placed upon the mathematical model for the floating magnets is an extremely important one. We claim only to have an equation that "fits" the experimental curve for small loads and relatively large separation distances. We may describe this restriction by saying that the domain of separation distances (for the converse relation) must be restricted. In a previous section the domain of the simple lens was all of the positive numbers. In the present experiment we cannot use the whole set of positive numbers. The domain of the present relation is governed by the ability of this function to follow the behavior of the magnets.

## Exercise 8

1. In the Floating Magnet experiment we obtained the relation

$$\ell = m(\frac{1}{s} - c) .$$

Algebraically obtain the converse of this relation. What separation distance does it predict for zero load?

2. For a limited domain, the floating magnet function was found to be

$$\ell = m(\frac{1}{s} - c) .$$

What is the unit of m? the unit of c?

3. Sketch roughly the graph of $y = \frac{k}{x}$ for $k < 0$.

4. A particular reciprocal relation is $y = \frac{1}{x}$ . Find the elements in the range that correspond to the following elements in the domain: $10^{-6}$, $10^{-4}$, $10^{-2}$, 1, $10^{2}$, $10^{4}$, $10^{6}$.

5. For the relation of the previous exercise, find the elements in the domain of the relation that correspond to the following elements in the range: $10^{-6}$, $10^{-4}$ $10^{-2}$, 1, $10^{2}$, $10^{4}$, $10^{6}$ .

6. Locate the x and y intercepts for the relation $y = \frac{k}{x}$ for $k > 0$.

## 5.14 Curve Fitting

Let us restate the procedure we used in the previous section to find a mathematical model for the magnet relation. We found that a mathematical model could be used to represent the results of an experiment *if* the domain of this mathematical relation was suitably restricted. Graphically we see that the curve for the model and the curve representing the data follow along together for a while, but soon their paths separate. We might say that the curve of the mathematical model "fits" the experimental curve in one region, but not in others.

Suppose that we had been interested in finding an equation that would accurately describe the behavior of the magnets for large loads and small separations rather than for small loads and large separations. Our previous model would be a poor one. But do you suppose it might be possible to "fit" a reciprocal relation to the experimental curve so that the situation for large loads would be described?

The best way to answer this question is to go back to your graph of the $(\frac{1}{s}, \ell)$ relation. See Figure 21. We drew a best straight line through the first four points before. We could, however, draw another straight line through the last three points. These points seem to line up fairly well. This would give us a new equation like the one obtained previously, but with different values of the constants. If now we were to calculate (s, ) points from this model and graph them, we would expect a "fit" to the experimental data along a quite different section of the experimental graph. We would now have to impose new restrictions upon the domain of the mathematical representation. It is important to note, however, that this model may be just as good in its domain and range as the first model was for small loads and large magnet separations.

In general, we are able to fit a reciprocal relation to the experimental relation for the floating magnets over any restricted part of the experimental curve we choose. This kind of procedure is called "curve fitting".

It should be pointed out that the reciprocal relation used to represent the behavior of the floating magnets is not the only reciprocal relation that might be used. We found that was a linear function of $\frac{1}{s}$ over a limited domain of $\frac{1}{s}$ values. We might also have tried to determine whether $\ell$ could be considered as a linear function of $\frac{1}{s^2}$, or even $\frac{1}{s^3}$. We know that whatever trial function we choose, the separation distance (s) must become smaller and smaller as the load ($\ell$) is increased. One of these new reciprocal relations might very well yield a much better fit to the experimental relation than the one used. By "better fit" is meant only that the graph of the mathematical equation might represent the experimental relation over a larger domain and range.

### Exercise 9

A beaker of water was heated on a hot plate. The temperature of the water was recorded every minute and the following data was obtained.

| Time (min) | Temp. ($^\circ$C) |
|---|---|
| 0 | 20 |
| 1 | 34 |
| 2 | 47 |
| 3 | 58 |
| 4 | 67 |
| 5 | 75 |
| 6 | 82 |
| 7 | 86 |
| 8 | 90 |

1. Graph the time-temperature relation. Over what range and domain would you say that the relation is a linear one?

2. Draw your best straight line to represent the time-temperature relation for a restricted time domain. Find the equation that represents this line.

3. Use the equation obtained in Exercise 2 to calculate temperatures for each of the nine time readings. What is the error in temperature prediction at times of 1 min; 4 min; 7 min ?

4. In the Floating Magnet experiment you made a graph of the reciprocal of the separation distance ($\frac{1}{s}$) along the horizontal axis and the load ($\ell$) along the vertical axis. Draw a best straight line through the points which represent loads of 120, 140 and 160 grams. Obtain the equation for this line. Calculate load values ($\ell$) from this equation, selecting 5 or 6 equally spaced s-values that will give loads in the range from 110 to 170 grams. Graph these calculated points and compare them to your original experimental points. Over what range of loads do you find a good "fit"?

## 5.15 Summary

In this chapter we have studied some characteristics of certain curves. We learned the meaning of the slope of a curve at a given point. This slope was found to have special physical significance as velocity. Our guess was verified to our satisfaction by comparing the measured velocity with one calculated from the graph.

From the data obtained in the Lens Experiment we were led to the study of the reciprocal function. By analyzing some properties of the hyperbola, it was found that the curve could be sketched through a study of its equation. While the curve drawn in this way was not an accurate physical model, a good approximation was obtained.

Finally, after working with the data from the Magnet Experiment, we learned that not all curves would fit into simple groups. This data presented us with the problem of a complex function from which we could arrive at only partial solutions.

This chapter, then, began our experience with the more complicated curves. As you continue to study mathematics, other more rigorous methods of obtaining some of the above information will be found.

# GLOSSARY

## Part III

ABSOLUTE VALUE -- The absolute value of a nonzero real number is the greater of that number and its opposite. The absolute value of zero is zero.

AMPLITUDE -- Maximum displacement on either side of an equilibrium position.

ANALOGY -- A form of mathematical inference based on the assumption that problems which have a similar appearance will have a similar treatment.

ANGLE OF INCLINATION -- The angle measured between the horizontal axis and the given line.

COMPARISON PROPERTY -- If a and b are real numbers, then exactly one of the following is true: $a < b$, $a = b$, $b < a$ .

CONJECTURE -- A conclusion reached without sufficient evidence for definite knowledge.

CONSTANT -- A constant is a number that remains unchanged during the course of a particular discussion.

CONTINUITY -- An uninterrupted succession in space of time.

CONVERSE -- Reversed in order, relation, or action.

COORDINATE PLANE -- The plane containing two perpendicular coordinate axes. Points in the coordinate plane are determined by ordered pairs of real numbers (coordinates).

DEFLECTION -- The amount of bend (as indicated by a pointer relative to a fixed scale).

DERIVATION -- Statements which show that a result is a necessary consequence of previously accepted statements.

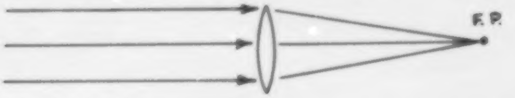
DISPLACE -- When a directed movement of a coordinate axis is made, we say that the axis is displaced.

DOMAIN -- The domain is the set of first elements of the ordered pairs in a relation or function.

EQUILIBRIUM -- The state of being in balance which occurs when the resultant of all outside forces acting on a body is zero.

FOCAL LENGTH -- The distance between a lens and the focal point.

FOCAL POINT -- The point at which a lens will cause parallel rays to converge.



FORCE -- Force is a physical concept which can be described loosely as the push or pull on an object.

FULCRUM -- The point of support and rotation of a seesaw or lever.

FUNCTION -- A function is a set of ordered pairs such that each element of the domain appears in one and only one ordered pair.

GENERATE -- To trace out mathematically by a moving point, line, or plane.

IMAGE DISTANCE -- The distance measured from some fixed point to the image. (In this text the focal point on the image side of the lens is taken as the fixed point.)

INTEGERS -- The set of counting numbers, zero, and the additive inverses of the counting numbers make up the set of integers.

INTERCEPT -- The point on a number line at which a second line meets it.

INTERPOLATE -- To find a value between two given values.

IRRATIONAL NUMBER -- A real number which cannot be expressed as the ratio of an integer to a counting number.

LINEAR -- Pertaining to straight lines.

MASS -- Mass is a fundamental property of a body. It is not the same as the weight of the body. On the earth's surface, the weight of an object is proportional to its mass.

MATHEMATICAL MODEL -- A mathematical relation which represents the physical model. In most situations it will be an equation.

MOMENT OF FORCE -- The moment of force is the turning effect of a force.

NEGATIVE INTEGERS -- The negatives of the set of counting numbers.

NEGATIVE REAL NUMBERS -- The set of real numbers associated with points to the left of zero on the number line, where the unit point is to the right of zero, is the set of negative real numbers.

OBJECT DISTANCE -- The distance measured from some fixed point to the object. In this text the focal point on the object side of the lens is taken as the fixed point.

ORDERED PAIR -- A set containing exactly two elements, (a,b), in which one element is recognized as the first element.

ORDERING PROPERTY FOR OPPOSITES -- For real numbers a and b, if $a < b$, then $-b < -a$ .

ORIENTATION -- Arranging correctly according to given facts or principles. Determining a position.

OSCILLATING -- Swinging from one extreme to another. To travel back and forth between points.

PERIOD -- The time interval between any event and the moment the same event occurs again is called the period.

PERPENDICULAR LINES -- Two lines which meet at right angles.

PHYSICAL MODEL -- A single curve on a graph of the set of points which best represents a collection of data. It is an idealization of the behavior of a physical system.

POSITIVE INTEGERS -- The set of counting numbers.

POSITIVE REAL NUMBERS -- The set of real numbers greater than zero. Usually represented by the points to the right of zero on the number line.

PROPERTY FOR OPPOSITES -- See Ordering Property for Opposites.

PROPERTY OF ORDER -- If a and b are two real numbers on the number line, and a is to the left of b, then $a < b$.

QUADRANT -- One of the four regions into which the coordinate axes divide the coordinate plane.

QUANTITATIVE -- Relating to or expressible in terms of quantity. Involving the measurement of quantity or amount.

RANGE -- The range is the set of second elements of the ordered pairs in a relation or function.

RATIONAL NUMBER -- A number which can be expressed as the ratio of an integer to a counting number.

REAL NUMBERS -- The set of all numbers associated with points on the number line. A number which can be represented by a finite or infinite decimal expansion.

RECIPROCAL -- The multiplicative inverse of a nonzero real number is called the reciprocal of the number. The reciprocal of a real number "a" ($a \neq 0$) is the number $\frac{1}{a}$ . Zero has no reciprocal.

RELATION -- A relation is a set of ordered pairs. When the pair (x,y) is in the set and we use R to represent the relation, we say that x R y is true.

REPELLED -- Tending to be forced away or apart.

SLOPE -- The slope measures the steepness of the inclination of a line. It is the ratio of the rise to the run.

SLOPE OF A CURVE -- The slope of a straight line which just touches the curve at a given point.

TERMINAL VELOCITY -- When the upward resistive force equals the downward gravitational pull on the object, terminal velocity has been reached.

TRANSLATION OF AXES -- Changing the coordinates of a set of points to coordinates referring to a new set of axes parallel to the original axes.

TRANSITIVE PROPERTY -- If a relation R has the property that whenever a R b and b R c are true statements then a R c is a true statement, we say that R has the transitive property.

UNIQUE -- Just one. Consisting on one and only one. Leading to one and only one solution.

VARIABLE -- A symbol which can be replaced by any member of a given set.

VELOCITY (CONSTANT) -- The slope of the line on a time-distance plot. It is given by $\frac{\text{distance}}{\text{time}}$ .